Asset Substitution and Structured Financing

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Abstract

This article shows how structured financing can be used to solve the asset substitution problem in a dynamic setting. Structuring induces the firm’s owner to optimally choose the first best operating strategy even though the owner’s value function might be locally convex (concave), which would ordinarily lead to overinvestment (underinvestment) in risky projects. This result is demonstrated in two different continuous time settings – one that is based on the risk shifting framework of Leland (1998) and one that generalizes the scaled return model of Green (1984). It is shown that the contractual nature of the structuring is a key determinant of the issuing firm’s dynamic asset volatility. Furthermore, unlike non-structured financing, the default (conversion) probability of a structured debt security may be increasing (decreasing) in the firm’s total assets. Structured securities are therefore hedge assets, which potentially explains the popularity of structured securities among investors and third-party issuers.
I. Introduction

A key insight of Jensen and Meckling (1976) is that external financing can alter the firm’s optimal operating strategy such that equityholders no longer maximize the firm’s total value. In the case of debt financing, the equityholders retain a convex claim on the firm’s assets and thus have an incentive to overinvest in risky assets after the debt is in place. The resulting reduction in total firm value is known as the agency cost of asset substitution. Green (1984) shows in a single period setting how this cost can be eliminated via capital structure design.\(^1\) Instead of using straight debt financing, the firm should use convertible debt financing and balance the convex and concave regions of the contract to create a security that is locally equity-like. However, as Hennessy and Tserlukevich (2007) demonstrate, convertible debt cannot eliminate the cost of asset substitution in a dynamic setting since equity is always risk-loving when the firm is near bankruptcy. Thus it remains an open question whether or not the asset substitution problem can be solved via capital structure design in a dynamic setting. This article resolves the question by showing how structured financing can induce the first best strategy even if the equityholders’ value function is locally convex or concave.

According to McCann and Cilia (1994), “structured securities are hybrids, having components of straight debt instruments and derivatives intertwined.” Thus structured finance employs the tools of financial engineering to facilitate the capital raising process. Structured securities are used in lieu of conventional securities, such as straight debt, and have exotic features that are included to satisfy the financing need of the issuer or the investment need of the buyer.\(^2\) Typical examples of structured securities include exchangeable bonds (Swieringa and Morse (1985), Barber (1993)), liquid yield option notes (McConnell and Schwartz (1986)), equity indexed notes (Chen and Sears (1990), Finnerty (1993), Gorton and Pennacchi (1993), Stoimenov and Wilkens (2005)), gold indexed securities (Chidambaran, Fernando, and Spindt (2001)), callable warrants (Schultz

\(^1\)Related early work includes Smith and Warner (1979), Barnea, Haugen, and Senbet (1980) and Haugen and Senbet (1981) who show how bond covenants, callable debt, and stock options, respectively, can mitigate the cost of asset substitution.

\(^2\)Although this article focuses on the issuer’s financing problem (i.e., the need to raise capital while simultaneously solving the asset substitution problem), it is shown below that structured securities are hedge assets that offer unique payoff patterns. Thus structured financing helps to satisfy the buyer’s investment need, which explains the interest from third-party issuers such as investment banks.
(1993)), reverse convertible debentures (Flannery (2005)), reverse exchangeable securities (Benet, Giannetti, and Pissaris (2005)), and death spiral convertibles (Hillion and Vermaelen (2004)).

This article focuses on a particular type of structured security in which the payoff-relevant features are correlated with the issuing firm’s assets. Table 1 lists several motivating examples. Each security in the table has a payoff that is tied to a contractually specified index. In turn, the index is correlated with the performance of the issuing firm’s assets. For example, a silver mining firm (Sunshine Mining) specifies a silver index in its bond indenture, a financial brokerage (Oppenheimer) specifies an index that is tied to the NYSE trading volume, an oil firm (Marathon Oil) specifies an oil index, etc. In other cases, the connection is less direct but equally relevant – the chemical firm Shin Etsu, which uses crude oil as an input to production, specifies an index that is inversely related to oil prices. Smithson and Chew (1992) and Chidambaran, Fernando, and Spindt (2001) argue that the securities in Table 1 can be viewed as bundled hedges. Bundling a hedge with a debt offering allows the issuing firm to credibly commit to the hedging strategy that maximizes the firm’s total value (Smith and Stulz (1985)).

This article identifies the particular bundled hedging strategy that should be used to solve the asset substitution problem. In other words, rather than study particular cases of bundling as in Chidambaran, Fernando, and Spindt (2001), this article derives an optimal bundling strategy. In the simpler dynamic settings considered here, the optimal bundling strategy completely eliminates the agency cost of dynamic asset substitution. In more complex settings, such as those that involve contracting constraints, the strategy mitigates but does not completely eliminate the agency cost. In both cases, the strategy involves structured securities whose payoff-relevant features are correlated with the firm’s total assets, as in Table 1. Furthermore, the correlation is implemented via a contractual structuring variable that serves the role of an index.

While straight debt with a fixed face value ordinarily leads to overinvestment, a structured debt contract in which the face value varies with the firm’s total assets mitigates the equityholders’ incentive to overinvest in risky assets. In the simpler dynamic settings considered here, the equityholders are induced to always choose the first best operating strategy and thus the agency

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3 This type of security is contained in the broader class of securities that is analyzed in Manso, Strulovici, and Tchisty (2006). However, unlike the current article, these authors do not examine the asset substitution problem.
cost of asset substitution is zero. A similar result holds if the equityholders' value function is locally concave, which would ordinarily lead to underinvestment, or if the value function has both convex and concave regions, which would ordinarily lead to dynamic risk shifting.

When a firm issues non-structured straight debt, the equityholders' (debtholders') exposure involves limited downside (upside) but substantial upside (downside). The oppositeness of these exposures creates an incentive for equityholders to deviate from the first best. Structuring the debt can eliminate this incentive by dynamically altering the upside and downside exposures of both equityholders and debtholders. To see this, consider a firm that is financed with equity and structured debt. The structured debt’s face value depends on the performance of the firm’s assets. In particular, suppose the face value’s elasticity with respect to the firm’s assets is bigger than one. As the assets increase (decrease) by 1%, the face value of the structured debt increases (decreases) by more than 1%, which in turn increases (decreases) the likelihood of default. If default occurs, the equity is worthless and the debtholders take ownership of the firm. Unlike the case of non-structured debt, the structured debtholders take ownership when the assets are at a relatively high level, which gives the debtholders upside exposure. If instead default does not occur, the equityholders retire the debt and retain ownership of the firm. However, unlike the case of non-structured debt, the equityholders retain ownership when the assets are at a relatively low level, which gives the equityholders downside exposure. By issuing a structured debt security with the proper dynamic elasticity, the equityholders' upside and downside exposures can be adjusted dynamically so that the equityholders always find it optimal to choose the first best operating strategy.

To illustrate the robustness of structured financing for solving the asset substitution problem, two different dynamic settings are considered. The first is a modified version of Leland’s (1998) risk shifting model.\footnote{Related (but not identical) risk-shifting models can be found in Carlson and Lazzar (2006), Chesney and Gibson-Asner (2001), Childs, Mauer, and Ott (2005), Decamps and Djembissi (2005), Ericsson (2000), Hennessy and Tseriukevich (2007), Mauer and Triantis (1994), and Subramanian (2005).} In this setting, the firm’s owner can costlessly switch back and forth between two different drift-diffusion pairs that govern the evolution of the firm’s total assets. Switching back and forth through time captures the dynamics of asset substitution, as described in Childs,
Mauer, and Ott (2005) and Hennessy and Tserlukevich (2007). Furthermore, since switching affects the asset drift, asset substitution can be value dissipating as in Mauer and Sarkar (2005). However, in contrast to the existing literature, this article uses a finite time horizon and does not rely on numerical methods to solve the problem.

The second setting is a generalized version of Green’s (1984) two-project model. Green’s (1984) analysis exhibits the following properties: (i) the first best strategy involves equal investment in the two projects; (ii) the optimal strategy in the presence of debt financing involves overinvestment in the riskier project; and (iii) a properly designed convertible debt contract can restore the first best strategy. This article constructs a continuous time model that replicates features (i) and (ii), but does not replicate (iii) for the reasons discussed in Hennessy and Tserlukevich (2007). It is then shown how structured financing can restore (iii). In other words, while a non-structured convertible debt contract solves the asset substitution problem in the single period model of Green (1984), a structured convertible debt contract (or some other structured contract, as shown later) is required to solve the asset substitution problem in a dynamic setting.

Childs, Mauer, and Ott (2005) and Ju and Ou-Yang (2005) show that the agency cost of asset substitution can be mitigated if a firm has sufficient financial flexibility to dynamically rebalance its short term debt level. Since the structured debt contracts in this article have contractual features (e.g., face values) that are time varying, structuring can be viewed as a substitute for dynamic debt rebalancing. However, in contrast to the dynamic debt rebalancing literature, all of the structured securities in this article are long term in nature. The firm finances its operations only once and thus does not require the flexibility to revisit the capital markets in the future for additional funding. Instead, the original financing stays in place but the payoff-relevant parameters change dynamically as the firm’s total assets change.

While the asset substitution problem was initially studied in a single period setting, the profession has recently taken an increased interest in studying the effects of dynamic asset substitution. For instance, Mello and Parsons (1992), Leland (1998), Ericsson (2000), Décamps and Djembissi (2005), Mauer and Sarkar (2005), and Parrino and Weisbach (1999) focus on quan-

tifying the agency cost of dynamic asset substitution. In contrast to these papers, the current article shows how structured financing can be used to resolve the bondholder-stockholder agency conflict in a dynamic setting. Unlike the existing literature, the main focus here is on solving the asset substitution problem rather than measuring its cost. The agency cost of dynamic asset substitution is equal to zero in the presence of a properly designed capital structure, at least for the simpler dynamic settings considered here.

The remainder of the article is organized as follows. Section II modifies the risk shifting framework of Leland (1998) and shows how a properly designed capital structure can eliminate the agency cost of asset substitution. The main result is summarized in Proposition 1 and explicit analytical solutions are presented in Tables 2-4 for a variety of parameter restrictions. Section III then generalizes Green’s (1984) model to a continuous time setting. Explicit analytical solutions for this setting are presented in Table 5. Section IV examines the characteristics of structured contracts and shows how the standard intuition concerning default probabilities, default costs, conversion probabilities, and expected yields may be incorrect for structured securities. Section V offers concluding remarks and directions for future research.

II. A Model With Two Risk-Return Choices

Consider a continuous time model in which uncertainty is represented by a one dimensional Brownian motion $z_t$ that is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Time is indexed by $t \in [0, T]$ and $\{\mathcal{F}_t : t \in [0, T]\}$ is the augmented filtration that is generated by $z_t$. The owner of a firm has access to a single risky project that requires an investment of $V_0 > 0$ at time 0. The project produces a payoff of $V_T$ at time $T$, where $V_t$ follows the process

\begin{equation}
    dV_t = \mu_t V_t dt + \sigma_t V_t dz_t
\end{equation}

for $t \in [0, T]$. The project does not produce intermediate cashflows nor does it require intermediate investments. If undertaken, the project in (1) will comprise the total assets of the firm.

The owner’s initial funds, denoted by $W$, satisfy $0 < W < V_0$. Thus to undertake the project, the owner must raise $V_0 - W$ via external financing. The final payoff $V_T$ is shared at time $T$ by the
owner and the external financier, where the sharing rule is determined by the capital structure put in place at time 0. As in Green (1984) and Green and Talmor (1986), it is assumed that there are no taxes, coupons, or debt amortization. Furthermore, unlike Black and Cox (1976) and Leland (1998), there is no possibility for the firm to reorganize prior to time $T$. While formally incorporating these features is possible, their omission simplifies the setting so that an explicit analytical solution is possible.

Two features of the model are worthy of comment. The first is why the firm would want to use debt financing as opposed to outside equity. In the absence of taxes, structured debt financing and outside equity are equally successful in solving the asset substitution problem. But once taxes are included, which are ignored only for simplicity, structured debt is the more attractive solution due to the tax deductibility of interest payments. In a world with taxes, structured debt maintains the tax advantages of non-structured debt but also has desirable equity-like features that allow the firm’s owner to solve the asset substitution problem. The second is why the firm would want to use structured debt as opposed to non-structured debt with a covenant requiring the owner to choose the first best. By using structured debt, the firm’s owner is able to design his residual claim so that it is optimal to choose the first best operating strategy for all $t \in [0,T]$. Since the owner has no incentive to deviate from the first best, the debtholders do not need to monitor the owner’s actions. On the other hand, by using non-structured debt with restrictive covenants, the owner’s incentive to expropriate value from the debtholders is not eliminated. Only with continuous monitoring, which may be costly, will the debtholders be assured that the owner follows the first best strategy for all $t \in [0,T]$. For these reasons, this article focuses on structured financing as the most attractive type of security for solving the asset substitution problem.

The owner’s operating discretion is modeled by allowing the owner to choose between two drift-diffusion pairs. At each $t$, the owner chooses $\{\mu_t, \sigma_t\}$ to be either $\{\mu_1, \sigma_1\}$ or $\{\mu_2, \sigma_2\}$, where $\mu_1, \sigma_1, \mu_2,$ and $\sigma_2$ are positive constants. The owner can switch back and forth between the pairs $\{\mu_1, \sigma_1\}$ and $\{\mu_2, \sigma_2\}$ without cost, where the switching rule is $\mathcal{F}_t$-measurable. Switching back

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6 This remains true in the case of zero coupon debt since the amortization of the original issue discount (OID) is treated as interest for tax purposes.
and forth allows the owner to engage in dynamic asset substitution.\footnote{Other types of switching models have also been used. For example, Mello and Parsons (1992) and Mauer and Triantis (1994) use a switching model for the firm’s decision to operate or shut down, while Childs, Mauer, and Ott (2005) use a switching model for the firm’s decision to exercise or reverse its investment in growth options.}

To capture the notion that the owner operates his firm in an economy that is populated by risk averse investors, the stochastic discount factor at time $t$ is

\begin{equation}
\frac{m_t}{m_T} = e^{-rt - \frac{1}{2} \lambda^2 t - \lambda z_t},
\end{equation}

where $r > 0$ is the constant instantaneous interest rate and $\lambda > 0$ is the constant instantaneous risk premium. Using (2), the time $t$ value of the firm is determined by evaluating the expectation $E_t \left[ \frac{m_T}{m_t} V_T \right]$, where $E_t [\cdot] = E [\cdot | \mathcal{F}_t]$ denotes expectation under the $P$ probability measure. For example, if the pair $\{\mu_1, \sigma_1\}$ is chosen for all $s \in [t, T]$, the firm’s value at time $t$ is

\begin{equation}
E_t \left[ \frac{m_T}{m_t} V_T \right] = V_t e^{(\mu_1 - \sigma_1 \lambda - r)(T-t)}.
\end{equation}

If instead the manager chooses the pair $\{\mu_2, \sigma_2\}$ for all $s \in [t, T]$, the time $t$ firm value is $V_t e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)}$. Thus the firm value is maximized by choosing the drift-diffusion pair at each $t$ that has the highest risk adjusted instantaneous return. In other words, the owner should always choose the pair that satisfies $\max [\mu_1 - \sigma_1 \lambda, \mu_2 - \sigma_2 \lambda]$. This is the first best strategy.

Since the initial firm value under the first best strategy is $V_0 e^{(\max [\mu_1 - \sigma_1 \lambda, \mu_2 - \sigma_2 \lambda] - r) T}$, the net present value (NPV) at time 0 is

\begin{equation}
V_0 e^{(\max [\mu_1 - \sigma_1 \lambda, \mu_2 - \sigma_2 \lambda] - r) T} - V_0.
\end{equation}

The NPV in (4) is positive if $\max [\mu_1 - \sigma_1 \lambda, \mu_2 - \sigma_2 \lambda] > r$. It is assumed that this inequality holds since otherwise the owner’s best course of action is to invest everything in the riskless asset.

Equation (2) can also be written as $m_t = e^{-rt} E_t \left[ \frac{dQ}{dP} \right]$, where $\frac{dQ}{dP} = e^{-\frac{1}{2} \lambda^2 t - \lambda z_t}$ defines a new probability measure $Q$ such that $dz_t = dz_t^Q - \lambda dt$ and $z_t^Q$ is a standard Brownian motion under
Q. The process for $V_t$ under the risk adjusted measure $Q$ is

$$dV_t = (\mu_t - \sigma_t \lambda) V_t dt + \sigma_t V_t dz_t^Q$$

and thus (3) can be rewritten as $E_t^Q [e^{-r(T-t)}V_T]$. This notation is used in the sequel.

A. Agency Cost of Asset Substitution

Following the literature, the agency cost of asset substitution is defined as the difference between the first best firm value and the firm value that is achieved in the presence of external financing. When external funds are used to finance the firm, the owner chooses a drift-diffusion pair at each $t$ to maximize his residual claim on the firm’s assets. If the owner’s optimal strategy in the presence of external financing differs from the first best strategy, the total firm value will be less than the first best value. Since the external financing is fairly priced at time 0, the reduction in total firm value at time 0 is borne entirely by the owner. This produces a positive agency cost of asset substitution and thus the owner fails to capture the full amount of the NPV in (4). On the other hand, if the owner’s choice of capital structure creates a residual claim whose optimizing strategy coincides with the first best strategy, the total firm value in the presence of external financing will coincide with the first best firm value. In this case, the owner captures the full amount of the NPV in (4) and the agency cost of asset substitution is zero. The next section shows how a zero agency cost can be achieved by using structured financing.

B. Structured External Financing Contracts

In this article, a structured security is an external financing contract whose payoff-relevant features (e.g., the face value or the conversion value) are allowed to covary with the firm’s assets.\(^8\)

To formally describe a structured security, let $\{\theta_t : t \in [0, T]\}$ be $\mathcal{F}_t$-measurable and define the

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\(^8\)Under this definition, a structured security can also be a hybrid security since it may contain both debt-like and equity-like features. However, not all hybrids are structured, e.g., a non-structured convertible bond. Structured securities are therefore a generalized class of security with hybrids and non-hybrids arising as special cases.
structuring variable $\xi_t$ as

$$
\xi_t = \exp \left[ -\frac{1}{2} \int_0^t \theta_s^2 ds + \int_0^t \theta_s dZ_s^Q \right].
$$

It is assumed that the functional form of $\theta_t$ is specified by the owner at time 0 and thereafter cannot be altered. In other words, $\{\theta_t : t \in [0, T]\}$ is part of the contract’s design, which is strictly enforceable. Note that $\xi_t$ and $V_t$ are correlated since $z_t^Q$ drives both processes. Thus $\xi_t$ serves the role of an index, as in Table 1.

The owner finances the firm by issuing a structured contract whose time $T$ payoff is

$$
F(V_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n),
$$

where $K_1, K_2, \ldots, K_n$ are constants. The contract $F$ is non-decreasing in $V_T$, has no intermediate cashflows (zero coupon), and is homogeneous of degree one in all of its arguments. If the owner specifies $\theta_t = 0$ for all $t$, then $\xi_T = 1$ and (7) reduces to a non-structured contract. At the other extreme, if the owner specifies a non-zero $\theta_t$ for all $t$, $\xi_T$ varies with the state of nature and $\xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n$ are correlated with the performance of the firm’s assets at time $T$.

For example, let $n = 2$ and suppose that $F$ is a structured convertible bond whose payoff is

$$
\min [V_T, \xi_T K_1] + \max [V_T - \xi_T K_2, 0].
$$

The face value $\xi_T K_1$ and the conversion value $\xi_T K_2$ are correlated with $V_T$. In the special case of $\theta_t = 0$ for all $t$, the structured convertible bond reduces to a non-structured convertible bond in which the bondholders take control of the firm upon conversion.9

By defining the structuring variable as in (6) and specifying an appropriate $\{\theta_t : t \in [0, T]\}$ at time 0, the owner can be induced to always choose the first best operating strategy. To see this, suppose the owner issues a structured security at time 0. Once the security is issued, the owner no longer has direct control over $\xi_t$ since it depends only on the path of $z_t^Q$, which is governed

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9 Non-structured convertible bonds are analyzed in Ingersoll (1977) and Brennan and Schwartz (1980, 1982). While Ingersoll (1977, Section 7) analyzes the case of a time varying conversion value, which is related to the term $\xi_T K_2$ in (8), he does not examine the asset substitution problem.
by nature, and on \( \{ \theta_t : t \in [0, T] \} \), which cannot be altered since it is part of the issued security’s design. However, due to the homogeneity of \( F \) and the form of (6), \( \xi_T \) can be factored out of the debt’s payoff and used as a Radon-Nikodym derivative to define a new probability measure (say, \( R \)) that is equivalent to the risk adjusted measure \( Q \). The structured security and the owner’s residual claim are valued by the market using \( R \). By cleverly specifying \( \{ \theta_t : t \in [0, T] \} \) at time 0, the owner can choose the particular \( R \) that induces the first best operating choice (e.g., \( \{ \mu_2, \sigma_2 \} \)) for all \( t \). Essentially, the owner designs the firm’s capital structure at \( t = 0 \) such that the incentive for dynamic asset substitution is eliminated for all \( t \in [0, T] \).

More specifically, \( \xi_t \) can be written as \( \xi_t = E_t^Q \left[ \frac{dR}{dQ} \right] \), where \( \frac{dR}{dQ} = \xi_T \) defines a new probability measure \( R \) such that \( dz_t^R = dz_t^R + \theta_t dt \) and \( z_t^R \) is a standard Brownian motion under \( R \). Thus the value of the owner’s residual claim is

\[
E_t^Q \left[ e^{-r(T-t)} (V_T - F(V_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)) \right]
= E_t^R \left[ \frac{\xi_t}{\xi_T} e^{-r(T-t)} (V_T - F(V_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)) \right]
= \xi_t E_t^R \left[ e^{-r(T-t)} (V_T \xi_T^{-1} - F(V_T \xi_T^{-1}; K_1, K_2, \ldots, K_n)) \right],
\]

which follows since \( F \) is homogeneous. The owner’s optimization problem is therefore

\[
\max_{\{\mu_t, \sigma_t\}} \xi_t E_t^R \left[ e^{-r(T-t)} (G_T - F(G_T; K_1, K_2, \ldots, K_n)) \right],
\]

where \( \{\mu_t, \sigma_t\} \) is either \( \{\mu_1, \sigma_1\} \) or \( \{\mu_2, \sigma_2\} \) at each \( t \) and where \( G_t = V_t \xi_t^{-1} \) follows the process

\[
dG_t = (\mu_t - \sigma_t \lambda) G_t dt + (\sigma_t - \theta_t) G_t dz_t^R
\]

and \( G_0 = V_0 \). Letting \( f(G_t, t) = E_t^R \left[ e^{-r(T-t)} (G_T - F(G_T; K_1, K_2, \ldots, K_n)) \right] \), Bellman’s principle of stochastic control can be used to restate the problem in (10) as

\[
0 = \max_{\{\mu_t, \sigma_t\}} \left[ \xi_t \left( \frac{\partial f}{\partial t} - rf + (\mu_t - \sigma_t \lambda) G_t \frac{\partial f}{\partial G} + \frac{1}{2} (\sigma_t - \theta_t)^2 G_t^2 \frac{\partial^2 f}{\partial G^2} \right) \right]
\]
subject to the terminal condition \( f(G_T, T) = G_T - F(G_T; K_1, K_2, \ldots, K_n) \). Two observations can be made about (12). First, since \( \xi_t \) factors out of the Bellman equation, the owner’s effective value function is simply \( f(G_t, t) \). Second, note that the terms multiplying \( \frac{\partial^2 f}{\partial G^2} \) include \( \theta_t \). This is a key property since a judicious choice of \( \theta_t \) can induce the owner to choose the first best strategy even if the sign of \( \frac{\partial^2 f}{\partial G^2} \) would ordinarily lead the owner to deviate from the first best.

For example, suppose \( \{\mu_2, \sigma_2\} \) is the first best pair and let \( \sigma_2 < \sigma_1 \). If the owner finances the firm with non-structured straight debt, \( \theta_t = 0 \) and the classical asset substitution problem arises. The owner’s equity stake resembles a call option on the firm’s assets and thus \( \frac{\partial^2 f}{\partial G^2} \) is positive. In this case, the owner has an incentive to abandon the first best pair \( \{\mu_2, \sigma_2\} \) in favor of the higher volatility, second best pair \( \{\mu_1, \sigma_1\} \). From (12), this incentive will be strongest when \( \frac{\partial^2 f}{\partial G^2} \) is large relative to \( \frac{\partial f}{\partial G} \). Choosing the second best pair produces a lower risk adjusted return, which multiplies \( \frac{\partial f}{\partial G} \), but this may be more than offset by the higher volatility, which multiplies \( \frac{\partial^2 f}{\partial G^2} \). To eliminate this incentive, the owner can credibly commit to the first best pair by issuing structured debt with \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \). To see that this value for \( \theta_t \) is appropriate, note that \( \frac{\partial^2 f}{\partial G^2} \) is also positive for (non-convertible) structured debt. Thus the owner can achieve the higher volatility \( \sigma_1 \) by choosing the lower volatility, first best pair \( \{\mu_2, \sigma_2\} \). This follows since \( \sigma_2 \) is the argument that maximizes \( (\sigma_t - (\sigma_1 + \sigma_2))^2 \), which in turn multiplies \( \frac{\partial^2 f}{\partial G^2} \) in (12). This simple example illustrates the logic that underlies structuring.\(^{10}\)

Given the above framework, three parametric cases are worth analyzing: (i) \( \mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 > \sigma_1 \); (ii) \( \mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 < \sigma_1 \); and (iii) \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 > \sigma_1 \). All three cases involve \( \sigma_2 \neq \sigma_1 \) since otherwise there is no risk shifting problem to study. Furthermore, the three cases are exhaustive since the omitted cases can be handled by reversing the subscripts 1 and 2. With these cases in mind, the article’s first result is presented in Proposition 1.

**Proposition 1.** Let \( \frac{\partial f}{\partial G} > 0 \). Then the agency cost of asset substitution is zero under each set of conditions listed below:

(i) let \( \mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 > \sigma_1 \). Furthermore, let \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial G^2} \geq 0 \) and let

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\(^{10}\)When \( \frac{\partial^2 f}{\partial G^2} > 0 \), it is tempting to argue that the owner should specify a very large value for \( \theta_t \). Although a large value induces the first best operating choice in the above example, it does not increase the value of the owner’s claim since the debt is fairly priced at time 0. The debt’s initial face value simply adjusts to reflect the large value for \( \theta_t \). This issue is revisited when structures that do not rely on the project parameters are discussed.
\( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \). Then the owner optimally chooses \( \{ \mu_2, \sigma_2 \} \) for all \( t \), which is the first best strategy;

(ii) let \( \mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 < \sigma_1 \). Furthermore, let \( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) and let \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \). Then the owner optimally chooses \( \{ \mu_2, \sigma_2 \} \) for all \( t \), which is the first best strategy;

(iii) let \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \) and \( \sigma_2 > \sigma_1 \). Since \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \), the owner always achieves the first best. If \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \) or if \( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \), the owner implements the first best by optimally choosing \( \{ \mu_1, \sigma_1 \} \) at time \( t \). If \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) or if \( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \), the owner implements the first best by optimally choosing \( \{ \mu_2, \sigma_2 \} \) at time \( t \).

The proof of the proposition is verified by examining the Bellman equation in (12). The owner’s choice of the drift-diffusion pair affects two terms in (12), the \( \frac{\partial f}{\partial \sigma} \) term and the \( \frac{\partial^2 f}{\partial \sigma^2} \) term. For case (i) of the proposition, the owner chooses \( \{ \mu_2, \sigma_2 \} \) at each \( t \), which is the first best strategy. This produces the highest risk adjusted return, which multiplies \( \frac{\partial f}{\partial \sigma} > 0 \). Furthermore, it maximizes \( \sigma_1^2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) and it minimizes \( (\sigma_t - (\sigma_1 + \sigma_2))^2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \).

For case (ii), the owner’s problem in (12) is again solved by choosing the first best pair \( \{ \mu_2, \sigma_2 \} \) at each \( t \). This produces the highest risk adjusted return, it maximizes \( (\sigma_t - (\sigma_1 + \sigma_2))^2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \), and it minimizes \( \sigma_t^2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \).

Since \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \) in case (iii), the agency cost of asset substitution is always zero.\(^{11}\)

However, the owner’s optimal strategy is affected by the sign of \( \frac{\partial^2 f}{\partial \sigma_1^2} \) and the value of \( \theta_t \). If \( \theta_t = 0 \) for all \( t \), the owner optimally chooses \( \{ \mu_2, \sigma_2 \} \) if \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) and optimally chooses \( \{ \mu_1, \sigma_1 \} \) if \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \). This is the case of pure dynamic risk shifting. Alternatively, if \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \), the owner optimally chooses \( \{ \mu_1, \sigma_1 \} \) if \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) and optimally chooses \( \{ \mu_2, \sigma_2 \} \) if \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \). This strategy also involves dynamic risk shifting but now the drift-diffusion pairs are reversed relative to the sign of \( \frac{\partial^2 f}{\partial \sigma_1^2} \). Dynamic risk shifting will also occur when \( \frac{\partial^2 f}{\partial \sigma_1^2} \) does not change signs but \( \theta_t \) changes back and forth between 0 and \( \sigma_1 + \sigma_2 \). On the other hand, structuring can be used to induce the owner to optimally commit to a particular drift-diffusion pair. If \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} \geq 0 \) and \( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial \sigma_1^2} < 0 \), the owner’s optimal strategy is to choose \( \{ \mu_2, \sigma_2 \} \) for

\(^{11}\)Using the terminology of Nachman and Noe (1996), asset substitution is value neutral when \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \). For the case of \( \mu_2 - \sigma_2 \lambda \neq \mu_1 - \sigma_1 \lambda \), value dissipating asset substitution can be avoided by using a capital structure that satisfies condition (i) or (ii) of Proposition 1.
all \( t \). Alternatively, if \( \theta_t = \sigma_1 + \sigma_2 \) when \( \frac{\partial^2 f}{\partial G^2} \geq 0 \) and \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial G^2} < 0 \), the owner’s optimal strategy is to choose \( \{ \mu_1, \sigma_1 \} \) for all \( t \). Many of these sub-cases are illustrated later.

C. Notation and a Preliminary Result

Note that the owner’s Bellman equation in (12) involves the instantaneous variance \( (\sigma_t - \theta_t)^2 \). If this variance is constant, both the owner’s residual claim and the value of the issued security can be expressed in terms of the Black and Scholes (1973) formula.\(^{12}\) Let \( C_{BS}(G_t, K, T - t, \delta, \sigma) \) denote the Black and Scholes European call option formula, where \( G_t \) is the underlying asset, \( K \) is the strike price, \( T - t \) is the time to expiration, \( \delta \) is the effective discount rate, and \( \sigma \) is the volatility parameter. These five inputs to the Black and Scholes formula will vary on a case by case basis depending on \( \theta_t \) and the owner’s optimal strategy. For example, the effective discount rate may be \( \mu_1 - \sigma_1 \lambda \) or \( \mu_2 - \sigma_2 \lambda \), the volatility parameter may be \( \sigma_1 \) or \( \sigma_2 \), and the underlying asset may be \( V_t \) (if \( \theta_t \) is always zero) or \( G_t \) (if \( \theta_t \) is non-zero).

If \( (\sigma_t - \theta_t)^2 \) is not constant, the Black and Scholes framework is no longer valid. In this case, given the values of \( \theta_t \) in Proposition 1, the instantaneous variance may periodically switch back and forth between \( \sigma_1^2 \) and \( \sigma_2^2 \). To handle this case, a preliminary result is presented that is used in the remainder of the article. Let \( \sigma_{\text{max}} = \max(\sigma_1, \sigma_2) \), \( \sigma_{\text{min}} = \min(\sigma_1, \sigma_2) \), and define \( g(t) \) as

\[
g(t) = Le^{-c(T-t)} - \frac{1}{2} \sigma_1 \sigma_2 (T-t),
\]

where \( c \) and \( L \) are positive constants. Furthermore, let \( D(G_t, t) \) denote the smooth continuous solution to the partial differential equation

\[
0 = \begin{cases} 
\frac{\partial D}{\partial t} - cD + cG_t \frac{\partial D}{\partial G} + \frac{1}{2} \sigma_{\text{max}}^2 G_t^2 \frac{\partial^2 D}{\partial G^2}, & G_t \leq g(t) \\
\frac{\partial D}{\partial t} - cD + cG_t \frac{\partial D}{\partial G} + \frac{1}{2} \sigma_{\text{min}}^2 G_t^2 \frac{\partial^2 D}{\partial G^2}, & G_t > g(t)
\end{cases}
\]

subject to the terminal condition \( D(G_T, T) = \max(0, G_T - K_1) - \max(0, G_T - K_2) \), where \( K_2 > K_1 \). A closed-form solution for \( D(G_t, t) \) is given in the Appendix. Under the restriction that \(^{12}\)For all of the capital structures considered, \( \mu_t - \sigma_t \lambda \) is constant along the optimal path and coincides with the first best risk adjusted return. Thus the sole determinant of whether or not the Black and Scholes (1973) framework is valid is the instantaneous variance.
\( K_1 = L e^{-b \sigma_{\text{max}}} \) and \( K_2 = L e^{b \sigma_{\text{min}}} \) for some constant \( b > 0 \),\(^\text{13}\) it is also shown in the Appendix that \( \frac{\partial^2 D}{\partial G^2} > 0 \) if \( G_t < g(t) \), \( \frac{\partial^2 D}{\partial G^2} < 0 \) if \( G_t > g(t) \), and \( \frac{\partial^2 D}{\partial G^2} = 0 \) if \( G_t = g(t) \). Thus \( g(t) \) is the boundary that separates the convex and concave regions of \( D(G_t, t) \). In the sequel, the function \( D(G_t, t) \) appears when the owner issues a structured convertible debt contract and \( (\sigma_t - \theta_t)^2 \) is not constant. To show an explicit dependence on the parameters, the function \( D(G_t, t) \) is written as

\[
(15) \quad D(G_t, t) = D(G_t, K_1, K_2, L, T - t, c, \sigma_{\text{max}}, \sigma_{\text{min}}).
\]

D. Capital Structures with Analytical Solutions

Using the results of Section II.C, Proposition 1 is illustrated for several simple capital structures. Case (i) is illustrated in Table 2. In Panel A of the table, the owner finances the firm by issuing non-structured debt, while in Panel B the owner instead issues structured convertible debt. In each panel, the structuring factor \( \theta_t \) is chosen so that the owner is induced to optimally choose the first best pair \( \{\mu_2, \sigma_2\} \) for all \( t \). For instance, since the owner’s residual claim in Panel A of Table 2 is convex, the owner has an incentive to choose the high volatility pair. Since this is also the first best pair, the structuring factor is set equal to zero. In Panel B, the owner’s residual claim is expressed in terms of the function \( D(G_t, t) \), which switches from convexity to concavity as \( G_t \) crosses \( g(t) \) from below. Thus, letting \( \theta_t = 0 \) when \( G_t \leq g(t) \) and \( \theta_t = \sigma_1 + \sigma_2 \) when \( G_t > g(t) \), the owner is induced to optimally choose \( \{\mu_2, \sigma_2\} \) for all \( t \).

The financing restrictions on \( K \) (Panel A of Table 2) and \( b \) (Panel B of Table 2) ensure that the issued security is fairly priced at time 0. Thus the owner captures the first best NPV in (4) and the agency cost of asset substitution is zero. In Panel B, the concave and convex regions of the structured convertible debt contract are balanced against one another at time 0 by choosing \( L \) such that \( g(0) = V_0 \). This restriction on \( L \) ensures that the owner’s value function is locally equity-like (i.e., a zero second derivative) at time 0 and is consistent with Green (1984).

Table 3 analyzes case (ii) of Proposition 1. In this case, the asset substitution problem is

\(^{13}\)This restriction is weak since for any given values of \( \sigma_{\text{min}}, \sigma_{\text{max}}, K_1, \) and \( K_2 > K_1 \), the constants \( L \) and \( b \) can always be chosen to solve the pair of equations \( K_1 = L e^{-b \sigma_{\text{max}}} \) and \( K_2 = L e^{b \sigma_{\text{min}}} \).
solved if the owner issues either structured debt (Panel A) or structured convertible debt (Panel B). As in Table 2, $\theta_t$ in each panel is specified such that the owner is induced to choose $\{\mu_2, \sigma_2\}$ for all $t$, which is the first best strategy. However, the structuring factors are different than their counterparts in Table 2 since now the first best pair $\{\mu_2, \sigma_2\}$ has the lower volatility.

If the firm is financed with structured debt (Panel A of Table 3), the owner’s residual claim is a call option on the firm’s assets with a strike price that is continuously adjusted to reflect the debt’s outstanding face value. To see this, recall that $G_t = V_t \xi_t^{-1}$ and note that $C_{BS}$ is homogeneous of degree one. Thus the owner’s value at time $t$ in Panel A of Table 3 can be rewritten as

$$e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)}C_{BS}(V_t, \xi_t K, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1).$$

The underlying asset is now $V_t$ and the strike price is $\xi_t K$, where the latter quantity represents the time $t$ face value of the structured debt. Note that structuring allows the owner to cherry pick the best feature of each drift-diffusion pair. The claim in (16) is valued using the higher risk adjusted return, $\mu_2 - \sigma_2 \lambda$, and the higher volatility, $\sigma_1$. Structuring allows the owner to credibly commit to $\{\mu_2, \sigma_2\}$ for all $t$, yet still enjoy the favorable valuation that is associated with the higher volatility. Of course there is a trade-off since $\xi_t K$ is positively correlated with $V_t$, which alters the set of states for which (16) finishes in the money relative to a call option with a constant strike price.

Table 4 illustrates case (iii) of Proposition 1. Although the first best NPV is always attained, the capital structure and the structuring factor have a profound effect on the owner’s optimal strategy. In turn, this affects the volatility pattern of the firm’s total assets. Four alternative convertible debt capital structures are shown in Table 4. Panel A illustrates the case of $\theta_t = 0$ (i.e., non-structured convertible debt). In this case, the owner optimally chooses the high volatility pair if $V_t \leq g(t)$ and otherwise chooses the low volatility pair. This is a case of pure dynamic risk shifting. The other panels show that structuring can induce the owner to optimally choose the opposite dynamic risk shifting strategy (Panel B), to always choose the high volatility pair (Panel C), or to always choose the low volatility pair (Panel D). Thus the type of external financing (e.g., convertible debt) is neither a good predictor of the owner’s optimal operating strategy nor a good
predictor of the volatility characteristics of the firm’s assets. The salient feature is how the debt is structured.

E. Operating Policy Agreement

It is worth noting that structured debt does not necessarily produce operating policy agreement between the owner and the debtholders. The reason for this is that when the owner’s value function is locally concave (convex), the debtholders’ value function is locally convex (concave). This follows since the total firm value always has a zero second derivative. Although structured debt induces the owner to optimally choose the first best operating policy, this choice may not be the preferred choice of the debtholders. Thus control rights are important – the solution to the asset substitution problem is contingent on the assumed control rights. If control rights are transferred from the owner to the debtholders without changing the capital structure, the asset substitution problem resurfaces. Only in special cases will structured debt simultaneously solve the asset substitution problem and align the operating policy choices of the owner and the debtholders. One instance of alignment is when \( \theta_t = \frac{1}{2} (\sigma_1 + \sigma_2) \). This is a knife edge case in which the owner’s choice of operating policy no longer depends on the sign of \( \frac{\partial^2 f}{\partial G^2} \) (see equation (12)). A second special case is discussed later in Section IV.F where it is shown how to structure debt to avoid costly default.

F. Additional Structuring Issues

The prior analysis involved two assumptions that are worthy of discussion. First, \( \xi_t \) in (6) was assumed to be perfectly instantaneously correlated with \( V_t \) even though some of the securities in Table 1 appear to use indices that are imperfectly correlated with the firm’s assets. Second, it was assumed that the firm could design its structured debt contract using the specific project parameters \( \sigma_1 \) and \( \sigma_2 \). Neither assumption is critical for the article’s main result – structured debt can eliminate the agency cost of asset substitution.

Concerning the first assumption, suppose (6) is generalized to take the form

\[
\tilde{\xi}_t = \exp \left[ -\frac{1}{2} \int_0^t \theta_s^2 ds + \int_0^t \theta_s dz_s^Q - \frac{1}{2} \int_0^t \beta_s^2 ds + \int_0^t \beta_s dw_s^Q \right],
\]
where \( \{ \beta_t : t \in [0, T] \} \) is exogenous and the Brownian motion \( w_t^Q \) is independent of \( z_t^Q \). Thus \( \tilde{\xi}_t \) is imperfectly correlated with the firm’s assets in (5). If the firm structures its debt using \( \tilde{\xi}_t \) instead of \( \xi_t \), the owner’s Bellman equation in (12) remains the same with the exception of the \( \frac{\partial^2 f}{\partial G^2} \) term. In particular, the quantity \((\sigma_t - \theta_t)^2\) must be replaced with \( \beta_t^2 + (\sigma_t - \theta_t)^2 \). This has no impact on the ability of structured debt to solve the asset substitution problem. The owner is still able to specify \( \{ \theta_t : t \in [0, T] \} \) as part of the issued security’s design in order to induce the first best, as in Proposition 1. However, the variance is now higher by \( \beta_t^2 \), which impacts the market’s valuation of the structured security. In turn, this affects the debt’s initial face value via the financing constraint.

Concerning the second assumption, Proposition 1 can be generalized to allow for structures that do not depend explicitly on the project parameters. To see this, suppose \( \{ \mu_2, \sigma_2 \} \) is the first best pair and let \( \sigma_2 > \sigma_1 \). This corresponds to Table 2 and case (i) of Proposition 1. If \( \frac{\partial^2 f}{\partial G^2} \geq 0 \), then any value of \( \theta_t \) that is less than \( \frac{1}{2} (\sigma_1 + \sigma_2) \) will induce the owner to choose the first best. Since \( \theta_t \) is less than \( \frac{1}{2} (\sigma_1 + \sigma_2) \), the distance between \( \theta_t \) and the owner’s choice \( \sigma_t \) is maximized by choosing \( \sigma_t = \sigma_2 \). For simplicity, case (i) of Proposition 1 uses \( \theta_t = 0 \) when \( \frac{\partial^2 f}{\partial G^2} \geq 0 \). On the other hand, if \( \frac{\partial^2 f}{\partial G^2} < 0 \), then any value of \( \theta_t \) that is greater than \( \frac{1}{2} (\sigma_1 + \sigma_2) \) will induce the owner to choose the first best. This follows since the distance between \( \theta_t > \frac{1}{2} (\sigma_1 + \sigma_2) \) and the owner’s choice \( \sigma_t \) is minimized by choosing \( \sigma_t = \sigma_2 \). Thus the values of \( \theta_t = 0 \) and \( \theta_t = \sigma_1 + \sigma_2 \) that are used in case (i) of Proposition 1 are sufficient, but not necessary, for the result to be true. The owner can use any pair of constants as long as the smaller (larger) constant is less (greater) than the average volatility. Given a particular pair of constants, the face value of the structured debt is adjusted such that the debt is fairly valued at time 0. Thus the owner’s initial equity value is not affected by the chosen pair of constants. A similar argument can be used to generalize cases (ii) and (iii) of the proposition, which correspond to Tables 3 and 4, respectively.

The above generalization suggests that structured debt may also be useful for solving the asset substitution problem when there is uncertainty about the project parameters. For example, let \( \{ \mu_2, \sigma_2 \} \) be the first best pair, but suppose the owner does not know the value of \( \sigma_2 \). For simplicity, assume \( \sigma_2 \) takes values in the interval \([\sigma_{2L}, \sigma_{2H}]\), where \( \sigma_{2L} < \sigma_{2H} \). There are three
cases of interest, two of which can be solved with structuring. First suppose that \( \sigma_{2L} > \sigma_1 \), where \( \sigma_1 \) is known. To induce the first best choice in this case, \( \theta_t \) should be less than \( \frac{1}{2} (\sigma_1 + \sigma_{2L}) \) if \( \frac{\partial^2 f}{\partial \theta^2} \geq 0 \) and should be greater than \( \frac{1}{2} (\sigma_1 + \sigma_{2H}) \) if \( \frac{\partial^2 f}{\partial \theta^2} < 0 \). Thus the debt is structured by using a pair of constants for \( \theta_t \) such that the distance between the two constants is large enough to bracket all possible values of the average volatility. Next suppose that \( \sigma_1 > \sigma_{2H} \). This is similar to the prior case, except that now the two inequalities involving \( \frac{\partial^2 f}{\partial \theta^2} \) must be reversed. Lastly, suppose that \( \sigma_{2H} > \sigma_1 > \sigma_{2L} \). In this case it does not appear that structured debt can solve the asset substitution problem. Regardless of the value of \( \theta_t \) or the sign of \( \frac{\partial^2 f}{\partial \theta^2} \) (unless the sign is zero), the owner may prefer to deviate from the first best.


In Section II, the owner’s operating choice is limited to the pairs \( \{\mu_1, \sigma_1\} \) and \( \{\mu_2, \sigma_2\} \). Limiting the operating choice in this manner does not drive the article’s results. This is shown by first generalizing Green’s (1984) one-period model, which involves a continuum of possible operating choices, to a continuous time setting. It is then shown how structured financing induces the first best. Structured financing is therefore a robust tool for solving the asset substitution problem.

In Green (1984, p. 117), the firm’s assets are equal to

\[
(17) \quad k(I_1)[1 + R_1] + k(I_2)[1 + R_2],
\]

where \( k(\cdot) \) is a scale function, \( R_j \) is the rate of return on project \( j \), and \( I_j \) is the amount allocated to project \( j \). Green (1984) assumes that \( R_2 = R_1 + z \), where \( z \) is unpriced noise. The two projects have identical risk adjusted expected returns, but project 2 is riskier than project 1 due to \( z \).

To extend the model to a continuous time setting, consider two correlated risky projects, \( V_{1,t} \) and \( V_{2,t} \), whose evolutions are described by

\[
(18) \quad dV_{1,t} = (\mu - \sigma \lambda)V_{1,t}dt + \sigma V_{1,t}dz_{1,t}^Q
\]

\[
(19) \quad dV_{2,t} = (\mu - \sigma \lambda)V_{2,t}dt + \sigma V_{2,t}dz_{1,t}^Q + \nu V_{2,t}dz_{2,t}^Q,
\]
where \( z_{1,t}^Q \) and \( z_{2,t}^Q \) are independent Brownian motions under the risk adjusted measure \( Q \). In (18)-(19), \( \mu \) is the drift parameter under the true probability measure, \( \sigma \) and \( \nu \) are the volatility parameters, and \( \lambda \) is a risk premium.\(^{14}\) It is assumed that \( \mu - \sigma \lambda > r \), where \( r \) is the interest rate. In addition, note that only the risk associated with \( z_{1,t} \) is priced. Thus the two projects have identical risk adjusted instantaneous returns, but project 2 is exposed to the unpriced idiosyncratic risk \( z_{2,t} \). These assumptions are consistent with those in Green (1984, pp. 118-119).

Following (17), the projects’ returns are combined using a concave scale function \( k(\cdot) \).\(^{15}\) However, rather than work with the total return as in (17), it is convenient in the continuous time model to work instead with the instantaneous return. The firm’s total assets, denoted by \( X_t \), are assumed to follow the process

\[
\frac{dX_t}{X_t} = k(I_{1,t}) \frac{dV_{1,t}}{V_{1,t}} + k(I_{2,t}) \frac{dV_{2,t}}{V_{2,t}},
\]

where \( I_{j,t} \geq 0 \) for \( j = 1, 2 \) is the capital stock allocated to project \( j \) at time \( t \). The left-hand side of (20) is the instantaneous return on the firm’s total assets, while the right-hand side of (20) is the continuous time counterpart to (17).

In Green’s (1984) single period model, the capital allocation decision occurs only once, at the beginning of the period. In contrast, it is necessary in the continuous time model to allow the owner to allocate the firm’s capital dynamically at each \( t \). To achieve this, it is assumed that \( I_{1,t} + I_{2,t} = I \) for all \( t \), where \( I \) is the constant total capital stock. The owner purchases capital (such as a factory) at time 0 by spending an amount equal to \( I \). Assuming perfect divisibility, the owner must then choose how to allocate this capital for each \( t \in [0,T] \). The capital allocated to project \( j \) at time \( t \) is scaled using \( k(\cdot) \) and earns the instantaneous return \( dV_{j,t}/V_{j,t} \), as in (20). For simplicity, the capital is assumed to be non-depreciable over \( [0,T] \) and thus \( I \) is a constant.

\(^{14}\)The notation in this section only partially overlaps with that in Section II. For example, \( \lambda \) is still a constant risk premium, but \( \mathcal{F}_t \) is now taken to be the augmented filtration that is generated by \( z_{1,t}^Q \) and \( z_{2,t}^Q \). Furthermore, the volatility parameters are now \( \sigma \) and \( \nu \) instead of \( \sigma_1 \) and \( \sigma_2 \).

\(^{15}\)The function \( k(\cdot) \) satisfies \( k(0) = 0, k' > 0, k'' < 0, \lim_{I \to \infty} k(I) = \ell < \infty \), and \( \lim_{I \to 0} k'(I) = \infty \). These assumptions are consistent with those in Green (1984, p. 117).
Substituting (18)-(19) into (20), the process for the total assets under $Q$ is\(^{16}\)

\[
\frac{dX_t}{X_t} = (\mu - \sigma \lambda) [k(I_{1,t}) + k(I_{2,t})] dt + \sigma [k(I_{1,t}) + k(I_{2,t})] dZ^Q_{1,t} + \nu k(I_{2,t}) dZ^Q_{2,t},
\]

where it is assumed that $X_0 = I$. Since the projects in (18)-(19) do not produce intermediate cashflows, there are no intermediate cashflows associated with $X_t$. The owner simply invests $X_0 = I$ at time 0 and then operates the firm by allocating $I$ between $I_{1,t}$ and $I_{2,t}$ for all $t \in [0, T]$.

If external financing is used to finance a portion of the initial investment $I$, the final payoff $X_T$ is shared at time $T$ between the owner and the external financier. The sharing rule is determined by the external financing contract that is put in place at time 0.

A. First Best Strategy

The first best strategy is the one that maximizes the total firm value. This strategy corresponds to the case in which the owner's initial funds are equal to $I$ and thus external financing is not needed. The owner’s Bellman problem in this case is

\[
0 = \max_{\{I_{1,t}, I_{2,t}\}} \left[ \frac{\partial f}{\partial t} - rf + (\mu - \sigma \lambda) [k(I_{1,t}) + k(I_{2,t})] X_t \frac{\partial f}{\partial X} + \frac{1}{2} \left( \sigma^2 [k(I_{1,t}) + k(I_{2,t})]^2 + \nu^2 [k(I_{2,t})]^2 \right) X_t^2 \frac{\partial^2 f}{\partial X^2} \right],
\]

subject to the constraint $I_{1,t} + I_{2,t} = I$ and the terminal condition $f(X_T, T) = X_T$. Due to the terminal condition, the value function $f$ should have the property $\frac{\partial^2 f}{\partial X^2} = 0$. The first order condition is therefore

\[
0 = (\mu - \sigma \lambda) \left[ k'(I_{1,t}^*) - k'(I_{2,t}^*) \right] X_t \frac{\partial f}{\partial X},
\]

which has the solution $I_{1,t}^* = I_{2,t}^* = 0.5I$. Since $k'' < 0$, the second order condition is satisfied. Thus the first best strategy involves equal allocation to the two projects for all $t$. This result coincides with Green (1984, p. 120). Since $I_{1,t}^* = I_{2,t}^* = 0.5I$, the total firm value at time $t$ is

\[^{16}\text{Note that (21) depends only on } X_t \text{ and not on } V_{1,t} \text{ or } V_{2,t}. \text{ This is due to the assumption in (20). If an alternative assumption is used, such as } X_t = k(I_{1,t})V_{1,t} + k(I_{2,t})V_{2,t}, \text{ the process for } X_t \text{ depends explicitly on } V_{2,t}. \text{ In this case, it can be shown that structuring still solves the asset substitution problem, but the closed-form solutions to the problem, such as those in Table 5, are lost.}\]
\( X_t e^{2(\mu - \sigma \lambda)(T-t) - r(T-t)} \), which solves (22).

B. Non-Structured Debt Financing and Overinvestment

Now suppose the owner has initial funds equal to \( W \) where \( 0 < W < I \). Thus the owner raises \( I - W \) via external debt financing. If the owner issues a non-structured debt contract with final payoff \( \min \{ X_T, K \} \), the owner’s Bellman problem is

\[
0 = \max_{\{I_{1,t}, I_{2,t}\}} \left[ \frac{\partial f}{\partial t} - rf + (\mu - \sigma \lambda) [k(I_{1,t}) + k(I_{2,t})] X_t \frac{\partial f}{\partial X} \right] \]

subject to \( I_{1,t} + I_{2,t} = I \) and the terminal condition \( f(X_T, T) = \max \{ 0, X_T - K \} \). The first order condition implies that

\[
k'(\hat{I}_{2,t}) = \frac{\left( (\mu - \sigma \lambda) X_t \frac{\partial f}{\partial X} + \sigma^2 \left[ k(\hat{I}_{1,t}) + k(\hat{I}_{2,t}) \right] X_t^2 \frac{\partial^2 f}{\partial X^2} \right)}{k'(\hat{I}_{1,t})}
\]

Since the owner’s claim \( f \) is increasing and convex, \( \frac{\partial f}{\partial X} \) and \( \frac{\partial^2 f}{\partial X^2} \) are both positive. Thus the fraction on the right-hand side of (24) is less than one, implying that \( k'(\hat{I}_{2,t}) < k'(\hat{I}_{1,t}) \). Due to the concavity of \( k(\cdot) \) it is concluded that \( \hat{I}_{2,t} > \hat{I}_{1,t} \) for all \( t \). The owner overinvests in the riskier project relative to the first best strategy, which produces a strictly positive agency cost of asset substitution. This coincides with Proposition 1 in Green (1984, p. 124).

Suppose instead that the owner finances the firm by issuing non-structured convertible debt. In this case the terminal condition for \( f \) changes but expressions (23)-(24) are otherwise unaltered. Thus, as claimed by Hennessy and Tserlukevich (2007), non-structured convertible debt does not restore the first best strategy. The owner can choose the parameters of the convertible bond so that \( \frac{\partial^2 f}{\partial X^2} \approx 0 \) at time 0, which implies equal initial allocation to the two projects. However, for any \( t > 0 \), it may be the case that \( \frac{\partial^2 f}{\partial X^2} > 0 \), which leads to overinvestment in the riskier project, or it may be the case that \( \frac{\partial^2 f}{\partial X^2} < 0 \), which leads to underinvestment in the riskier project. Thus there is a strict departure between the continuous time framework and the one-period model of
C. Structured External Financing Contracts

Define the structuring variable $\xi_t$ as

$$
\xi_t = \exp \left[ -\frac{1}{2} \int_0^t \theta_{1,s}^2 ds - \frac{1}{2} \int_0^t \theta_{2,s}^2 ds + \int_0^t \theta_{1,s}d\xi_{1,s}^Q + \int_0^t \theta_{2,s}d\xi_{2,s}^Q \right],
$$

where $\theta_{1,t}$ and $\theta_{2,t}$ are $\mathcal{F}_t$-measurable structuring factors whose functional forms are chosen at time 0 as part of the issued security’s design. The owner finances $I - W$ at time 0 by issuing a contract whose payoff at time $T$ is $F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)$. Like (7), the contract $F$ has no intermediate cashflows, it is non-decreasing in $X_T$, and it is homogeneous of degree one in all of its arguments.

Note that $\xi_t$ can be written as $\xi_t = E_t^Q \left[ \frac{dR}{dQ} \right]$, where $\frac{dR}{dQ} = \xi_T$ defines a new probability measure $R$ such that $dz_{1,t}^Q = dz_{1,t}^R + \theta_{1,t} dt$, $dz_{2,t}^Q = dz_{2,t}^R + \theta_{2,t} dt$, and $z_{1,t}^R$ and $z_{2,t}^R$ are independent Brownian motions under $R$. Thus the value of the owner’s residual claim is

$$
E_t^Q \left[ e^{-r(T-t)} (X_T - F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)) \right] = E_t^R \left[ \frac{\xi_t}{\xi_T} e^{-r(T-t)} (X_T - F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)) \right] = \xi_t E_t^R \left[ e^{-r(T-t)} (X_T \xi_T^{-1} - F(X_T \xi_T^{-1}; K_1, K_2, \ldots, K_n)) \right],
$$

which follows since $F$ is homogeneous. The owner’s optimization problem is therefore

$$
\max_{\{I_{1,t}, I_{2,t}\}} \xi_t E_t^R \left[ e^{-r(T-t)} (G_T - F(G_T; K_1, K_2, \ldots, K_n)) \right],
$$

subject to $I_{1,t} + I_{2,t} = I$ for all $t$. The process for $G_t = X_t \xi_t^{-1}$ is

$$
\frac{dG_t}{G_t} = (\mu - \sigma \lambda) [k(I_{1,t}) + k(I_{2,t})] dt + [\sigma (k(I_{1,t}) + k(I_{2,t})) - \theta_{1,t}] dz_{1,t}^R + [\nu k(I_{2,t}) - \theta_{2,t}] dz_{2,t}^R
$$

where $G_0 = X_0$. Letting $f(G_t, t) = E_t^R \left[ e^{-r(T-t)} (G_T - F(G_T; K_1, K_2, \ldots, K_n)) \right]$, Bellman’s principle of stochastic control is used to restate (27) as
subject to $I_{1,t} + I_{2,t} = I$ and the terminal condition $f(G_T,T) = G_T - F(G_T; K_1, K_2, \ldots, K_n)$. Note that (29) differs from (23) due to the presence of $\xi_t$, $\theta_{1,t}$, and $\theta_{2,t}$. The factors $\theta_{1,t}$ and $\theta_{2,t}$ will play an important role in solving the asset substitution problem.

1. Optimal Structuring with Black-Scholes (1973) Pricing

The first order condition to the Bellman problem in (29) can be written as

$$
0 = \max_{\{I_{1,t}, I_{2,t}\}} \left[ \xi_t \left( \frac{\partial f}{\partial t} - rf + (\mu - \sigma \lambda) [k(I_{1,t}) + k(I_{2,t})] G_t \frac{\partial f}{\partial I_{1,t}} + \frac{1}{2} \left[ (\sigma (k(I_{1,t}) + k(I_{2,t})) - \theta_{1,t})^2 + [\nu k(I_{2,t}) - \theta_{2,t}]^2 \right] G_t^2 \frac{\partial^2 f}{\partial I_{1,t}^2} \right) \right],
$$

where $f_G = \frac{\partial f}{\partial G}$ and $f_{GG} = \frac{\partial^2 f}{\partial G^2}$. If the owner specifies $\theta_{2,t} = \nu k(0.5I)$ for all $t$, the solution to (30) is $I_{1,t}^* = I_{2,t}^* = 0.5I$, which is the first best strategy. Note that the volatility coefficient of the idiosyncratic risk in (28) vanishes along the optimal path when $\theta_{2,t} = \nu k(0.5I)$. While the owner deviates from the first best in the absence of structuring, as shown in Section III.B, the structured contract eliminates the idiosyncratic risk from the owner’s problem and induces the owner to choose the first best. The second order condition depends on the sign of

$$
2(\mu - \sigma \lambda) k''(0.5I) G_t f_G + \left[ \nu^2 (k'(0.5I))^2 + 4\sigma^2 k(0.5I) k''(0.5I) - 2\sigma k''(0.5I) \theta_{1,t} \right] G_t^2 f_{GG},
$$

where $\theta_{2,t} = \nu k(0.5I)$ has been used. By letting

$$
\theta_{1,t} = \frac{\nu^2 (k'(0.5I))^2 + 4\sigma^2 k(0.5I) k''(0.5I)}{2\sigma k''(0.5I)} = 2\sigma k(0.5I) + \frac{\nu^2 (k'(0.5I))^2}{2\sigma k''(0.5I)},
$$

the terms that multiply $f_{GG}$ in (31) vanish. Thus the second order condition for a maximum is satisfied regardless of the sign of $f_{GG}$. Furthermore, when $\theta_{2,t} = \nu k(0.5I)$ and $\theta_{1,t}$ is given by (32), (28) has constant volatility, which implies the Black-Scholes pricing framework is valid.
However, alternative forms of $\theta_{1,t}$ may destroy the Black-Scholes pricing, as shown next.

2. Optimal Structuring with Risk Shifting

Suppose the owner finances the firm by issuing a structured convertible bond whose payoff is
\[ \min[X_T, \xi_T K_1] + \max[0, X_T - \xi_T K_2]. \]
From (30), the first order condition is satisfied by letting $\theta_{2,t} = \nu k(0.5I)$ and $I_{1,t} = I_{2,t} = 0.5I$ for all $t$, which is the first best strategy. Although $\theta_{1,t}$ in (32) induces Black-Scholes pricing, it is possible to specify a different $\theta_{1,t}$ that still satisfies (31) but induces pricing that is consistent with risk shifting. To see this, note that since $\mu - \sigma \lambda > 0$, $k'' < 0$, and $\frac{\partial f}{\partial G} > 0$, the first term in (31) is always negative. However, the second term in (31) depends on the quantity
\[ \nu^2 (k'(0.5I))^2 + 4\sigma^2 k(0.5I)k''(0.5I). \]
If (33) is negative, the owner can structure the convertible bond using
\[ \theta_{1,t} = \begin{cases} 0 & G_t \leq g(t) \\ 2\sigma k(0.5I) + \frac{\nu^2 (k'(0.5I))^2}{2\sigma k''(0.5I)} & G_t > g(t) \end{cases}, \]
The reasoning is that since $G_t \leq g(t)$ corresponds to $\frac{\partial^2 f}{\partial G^2} \geq 0$, the owner can let $\theta_{1,t} = 0$ and still be assured that (31) is negative. However, $G_t > g(t)$ corresponds to $\frac{\partial^2 f}{\partial G^2} < 0$. In this case, letting $\theta_{1,t}$ equal (32) assures that the $\frac{\partial^2 f}{\partial G^2}$ terms in (31) vanish. This leads to risk shifting in (28).

On the other hand, if (33) is positive, the owner can structure the convertible bond using
\[ \theta_{1,t} = \begin{cases} 2\sigma k(0.5I) + \frac{\nu^2 (k'(0.5I))^2}{2\sigma k''(0.5I)} & G_t \leq g(t) \\ 0 & G_t > g(t) \end{cases}, \]
where the reasoning is similar to the prior case. In (34)-(35), the switching boundary $g(t)$ is
\[ g(t) = Le^{-2(\mu - \sigma \lambda)k(0.5I)(T-t)} - \frac{1}{2} \sigma_{\min} \sigma_{\max} (T-t), \]
where \( L > 0 \), \( \sigma_{\text{min}} = \min \left[ 2\sigma k(0.5I), -\frac{\nu^2(k'(0.5I))^2}{2\sigma k'(0.5I)} \right] \), and \( \sigma_{\text{max}} = \max \left[ 2\sigma k(0.5I), -\frac{\nu^2(k'(0.5I))^2}{2\sigma k''(0.5I)} \right] \). Note that both \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) are positive since \( k'' < 0 \).

Since (33) depends on \( k(\cdot) \), the owner’s choice of how to structure the convertible bond depends on the firm’s scale function. Two firms that face the same pair of projects in (18)-(19) and are otherwise identical except for their scale functions may have different capital structures. Expression (33) may be positive for one firm and negative for the other.

Panel A of Table 5 shows the case when (33) is negative and \( \theta_{1,t} \) is given by (34). The negative value of (33) implies that \( \sigma_{\text{min}} = -\frac{\nu^2(k'(0.5I))^2}{2\sigma k'(0.5I)} \) and \( \sigma_{\text{max}} = 2\sigma k(0.5I) \). Using these values of \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) to construct \( K_1 = Le^{-b\sigma_{\text{max}}} \) and \( K_2 = Le^{b\sigma_{\text{min}}} \), the value of the owner’s residual claim is expressed in terms of the function \( D \) in (15). At time 0, the concave and convex regions of the structured convertible bond are balanced against one another by choosing \( L \) in (36) such that \( g(0) = G_0 \). Thus \( \frac{\partial f}{\partial G} = 0 \) at time 0. Given this value for \( L \), the financing restriction on the parameter \( b \) guarantees that the issued security is fairly priced at time 0. The owner captures the first best NPV and the agency cost of external financing is zero.

Panel B of Table 5 shows the case when (33) is positive and \( \theta_{1,t} \) is given by (35). The positive value of (33) implies that \( \sigma_{\text{min}} = 2\sigma k(0.5I) \) and \( \sigma_{\text{max}} = -\frac{\nu^2(k'(0.5I))^2}{2\sigma k''(0.5I)} \). Letting \( K_1 = Le^{-b\sigma_{\text{max}}} \) and \( K_2 = Le^{b\sigma_{\text{min}}} \), the owner’s claim is again expressed in terms of \( D \). However, this differs from Panel A since \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) are reversed. At time 0, the concave and convex regions are again balanced against one another by choosing \( L \) such that \( g(0) = G_0 \), and the financing restriction on \( b \) guarantees that the owner captures the first best NPV.

D. Discussion

From (30), specifying \( \theta_{2,t} = \nu k(0.5I) \) is not only sufficient but also necessary for achieving the first best. This is different than the result in Section II.F since now the structuring factor must depend on the project-specific parameter \( \nu \) and the scale function \( k(\cdot) \) if the owner wants to solve the asset substitution problem. It is therefore worth exploring how the agency cost of asset substitution is impacted when debt structures that rely on \( \nu \) and \( k(\cdot) \) are precluded.

If \( \theta_{2,t} \neq \nu k(0.5I) \), (30) shows that the first best is nearly achieved when \( f_{GG} \approx 0 \). On the
other hand, the owner’s investment strategy may deviate significantly from the first best when the magnitude of \( f_{GG} \) is large. The worst case scenario is characterized by taking the limit as \( f_{GG} \rightarrow \pm \infty \). In this case, (30) reduces to

\[
(37) \quad \frac{k'(I_{2,t})}{k'(I_{1,t})} = \frac{\sigma (\sigma [k(I_{1,t}) + k(I_{2,t})] - \theta_{1,t})}{\sigma (\sigma [k(I_{1,t}) + k(I_{2,t})] - \theta_{1,t}) + \nu (\nu k(I_{2,t}) - \theta_{2,t})}.
\]

Note that the right-hand side of (37) tends to 1 as the magnitude of \( \theta_{1,t} \) becomes large. Thus even if the owner is precluded from using \( \nu \) and \( k(\cdot) \) to structure the debt, the agency cost of asset substitution can be mitigated by choosing a large enough value for \( \theta_{1,t} \).

This is illustrated in Figure 1, which uses the scale function \( k(z) = z^\gamma \). The left graph illustrates the agency cost of asset substitution for three different values of \( \gamma \).\(^{17}\) For example, if \( \gamma = 0.5 \) and \( I_{1,t} = 0 \) for all \( t \), the agency cost is around 10\%, which is economically significant. The right graph shows that the cost can be mitigated by structuring the debt with a large value for \( \theta_{1,t} \).\(^{18}\) For example, if the magnitude of \( \theta_{1,t} \) is close to 100, \( I_{1,t} \approx 50 \). According to the left graph, the agency cost is small if \( I_{1,t} \approx 50 \); this is very close to the first best since \( I = 100 \). Thus the ability of structured debt to mitigate the agency cost of asset substitution is similar to that of dynamic debt rebalancing (see Childs, Mauer, and Ott (2005) and Ju and Ou-Yang (2005)).

IV. Characteristics of Structured Debt

This section contrasts the characteristics of structured debt with those of non-structured debt. The focus is on debt payoffs, default probabilities, expected yields, default costs, conversion probabilities, and conversion values.

A. Structured vs. Non-structured Payoffs

The payoffs for structured and non-structured debt are illustrated in Figure 2. Moving clockwise from the top left, the figure illustrates non-structured debt (as in Panel A of Table 2), non-structured convertible debt (Panel A of Table 4), structured convertible debt (Panel B of

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\(^{17}\) The agency cost in the left graph is defined as the first best firm value divided by the second best firm value, minus one. This coincides with the definition in Childs, Mauer, and Ott (2005).

\(^{18}\) The sign of \( \theta_{1,t} \) (positive or negative) is chosen to satisfy the second order condition.
Table 4), and structured debt (Panel A of Table 3). All of the graphs use $V_0 = 100$ and $W = 50$. Thus the owner raises 50 via debt financing.

The non-structured debt in the top left graph of Figure 2 is standard. For high asset values, the debtholders receive the promised face value $K$. For low asset values, the debtholders receive less than $K$ but take ownership of the firm. This standard payoff can be contrasted with the structured debt payoff $\min[V_T, \xi_T K]$ that is shown in the bottom left graph. Since the elasticity of $\xi_T$ with respect to $V_T$ is greater than one, default occurs at high asset values rather than low asset values. For low asset values, the firm is not in financial distress. For this region, the structured debt payoff is increasing and convex. Default first occurs when $V_T$ is about 230, which corresponds to the kink in the graph. For higher asset values, the graph is linear since the debtholders take ownership of the firm. The graph shows that structured debt is remarkably equity-like. Debtholders have upside participation and take control of the firm in good states. This allows the owner to eliminate the agency cost of asset substitution. However, the owner now has downside exposure and maintains control of the firm in bad states.

The top right graph shows non-structured convertible debt. For low asset values, the firm defaults and the debtholders receive less than the debt’s face value; for high asset values, the debtholders convert their debt into equity and take control of the firm. This standard payoff can be contrasted with the structured convertible debt payoff in the bottom right graph. Like structured debt, the elasticity of $\xi_T$ with respect to $V_T$ is greater than one. Thus default occurs at high asset values while conversion occurs at low asset values. The payoff due to conversion corresponds to the hump in the graph, which occurs when $V_T$ is less than about 50. By structuring the convertible debt to eliminate the agency cost of asset substitution, the debt’s payoff characteristics (i.e., default and conversion) are reversed relative to those of non-structured debt.

B. Default and Conversion Probabilities

The contrast between structured and non-structured financing is also seen by examining the sensitivity of the default and conversion probabilities to movements in the firm’s total assets.
First consider the non-structured debt contract in Panel A of Table 2. The default probability is

\[ E_t \left[ 1_{\{V_T < K\}} \right] = N \left( \frac{\log \left( \frac{K}{V_t} \right) - (\mu_2 - \frac{1}{2}\sigma_2^2) (T - t)}{\sigma_2 \sqrt{T - t}} \right) , \]

where \( 1_{\{\cdot\}} \) is the indicator function for the event in curly brackets and \( N(\cdot) \) is the standard normal cumulative distribution function. Since the derivative of (38) with respect to \( V_t \) is negative, the default probability decreases as the firm’s assets increase.

The result for structured financing is remarkably different since \( \xi_t \) is correlated with \( V_t \). To determine the sensitivity of the default probability, both the direct effect due to \( V_t \) and the indirect effect due to \( \xi_t \) must be analyzed. Consider the structured debt contract in Panel A of Table 3. Since \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \), \( \xi_t \) is path independent and can be written in terms of \( V_t \) as

\[ \xi_t = \left( \frac{V_t}{V_0} \right)^{1+\frac{\sigma_1}{\sigma_2}} e^{-\frac{1}{2}(\sigma_1+\sigma_2)^2t+(\sigma_1+\sigma_2)\lambda t-(\mu_2-\frac{1}{2}\sigma_2^2)(1+\frac{\sigma_1}{\sigma_2})t}. \]

The default probability is \( E_t \left[ 1_{\{V_T < \xi_t K\}} \right] \), which is rewritten as

\[ E_t \left[ 1_{\{G_T < K\}} \right] = N \left( \frac{\log \left( \frac{K}{G_t} \right) - (\mu_2 - \frac{1}{2}\sigma_2^2 + (\sigma_1 - \lambda)(\sigma_1 + \sigma_2)) (T - t)}{\sigma_1 \sqrt{T - t}} \right) . \]

Using \( G_t = V_t \xi_t^{-1} \), the derivative of (40) with respect to \( V_t \), including both the direct and indirect effects, is positive. This occurs since the elasticity is \( \frac{\partial \xi_t}{\partial V_t} \frac{V_t}{\xi_t} = 1 + \frac{\sigma_1}{\sigma_2} > 1 \), which can be seen from (39). When \( V_t \) increases by 1%, the face value of the debt increases by more than 1%. Contrary to the standard intuition, a higher asset level does not produce a lower default probability.

For the case of structured convertible debt, the analysis is considerably more complex due to the presence of the boundary \( g(t) \). In this case \( \xi_t \) may be path dependent since it may depend on the amount of time that \( G_t \) spends in the vicinity of \( g(t) \), as measured by the local time.\(^{20}\)

\(^{19}\)The default probabilities in this section are calculated under the true (i.e., \( P \)) probability measure. Calculating these probabilities under \( Q \) instead of \( P \) does not alter the conclusions.

\(^{20}\)The local time of a continuous stochastic process is defined in Karatzas and Shreve (1991, p. 218). For the structured convertible debt contracts in Figures 2 and 3, it is assumed that that the local time term that affects \( \xi_t \) is zero, thus allowing these figures to be constructed without referring to a particular path. This assumption does not impact the qualitative nature of these figures.
Furthermore, the elasticity of $\xi_t$ with respect to $V_t$ may change as $G_t$ crosses $g(t)$. For example, consider Panel B of Table 4. In this case, regardless of whether $G_t$ is above or below $g(t)$, the elasticity is always greater than one. Thus the prior conclusion that the default probability is increasing in $V_t$ also applies to structured convertible debt. In addition, the conversion probability is decreasing in $V_t$. However, from Panels C and D of Table 4, structuring may produce a mix of results since the elasticity may be greater than one on one side of $g(t)$ and may be equal to zero on the other side.

For the generalized version of Green’s (1984) model in Section III, the implications of structured financing for the default probability are equally interesting. For example, suppose (33) is negative and the owner finances the firm by issuing a structured debt contract with payoff

$$\min\{X_T, \xi_T K\},$$

where $\theta_{1,t} = 0$ and $\theta_{2,t} = \nu k(0.5I)$ for all $t$. Although this capital structure is not shown in the tables, it is verified from (30)-(31) that the first best is achieved. The default probability is

$$E_t [1_{\{X_T < \xi_T K\}}] = E_t [1_{\{G_T < K\}}] = N\left(\frac{\log \left(\frac{K}{G_T}\right) - 2k(0.5I) \left[\mu - \sigma^2 k(0.5I)\right] (T - t)}{2\sigma k(0.5I)\sqrt{T - t}}\right).$$

Since $\theta_{1,t}$ and $\theta_{2,t}$ are constant, $\xi_t$ is path independent and can be written as

$$\xi_t = \left[\frac{V_{2,t} V_{1,0}}{V_{2,0} V_{1,t}}\right]^{k(0.5I)} e^{-\frac{1}{2}v^2 k^2 (0.5I)t + \frac{1}{2} v^2 k(0.5I)t},$$

Furthermore, note that $X_t$ can be written as

$$X_t = X_0 \left[\frac{V_{2,t} V_{1,0}}{V_{2,0} V_{1,t}}\right]^{k(0.5I)} e^{-\frac{1}{2}v^2 k^2 (0.5I)t + \frac{1}{2} v^2 k(0.5I)t + \sigma^2 k(0.5I)t - 2\sigma^2 k^2 (0.5I)t}.$$ 

Recalling that $G_t = X_t \xi_t^{-1}$, expressions (42)-(43) imply that

$$G_t = X_0 \left[\frac{V_{1,t}}{V_{1,0}}\right]^{2k(0.5I)} e^{\sigma^2 k(0.5I)t - 2\sigma^2 k^2 (0.5I)t}.$$ 

Thus $G_t$ does not depend on the idiosyncratic risk $z_{2,t}$. This has important implications for the
default probability in (41). For example, consider a scenario in which $z_{2,t}$ increases but $z_{1,t}$ is unchanged. In this case, $V_{2,t}$ and $X_t$ both increase, but $V_{1,t}$ and $G_t$ are unchanged. Thus a change in the firm’s assets $X_t$ does not necessarily alter the firm’s default probability in (41).

Alternatively, consider a scenario in which a change in $z_{1,t}$ is offset by a change in $z_{2,t}$ such that $X_t$ is unchanged. Now the firm’s assets are unchanged, but the change in $z_{1,t}$ produces a change in both $V_{1,t}$ and (41). Thus the default probability may change even if the assets do not change.

A similar phenomenon arises for the structured convertible bonds in Table 5. However, the calculations are considerably more involved due to the switching boundary $g(t)$. But the central result in unaltered – in the presence of structured financing, the default (conversion) probability may no longer be a decreasing (increasing) function of the firm’s assets, which runs counter to the standard intuition.

C. Expected Yields

For the reasons discussed above, the expected yields of the structured and non-structured securities have remarkably different asset sensitivities. For the non-structured debt contract in Panel A of Table 2, the time $t$ expected yield is

$$\frac{K}{e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} \left[V_t - C_{BS}(V_t, K, T-t, \mu_2 - \sigma_2 \lambda, \sigma_2)\right]},$$

which equals the face value $K$ divided by the time $t$ debt value. It is easy to show that this yield is a decreasing function of $V_t$. In contrast, for the structured debt contract in Panel A of Table 3, the time $t$ expected yield is

$$\frac{E_t [\xi_T K]}{e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} \left[V_t - \xi_t C_{BS}(G_t, K, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1)\right]},$$

where the numerator is the expected face value and the denominator is the structured debt’s time $t$ value. Since $\theta_t = \sigma_1 + \sigma_2$ for all $t$, the expected face value is $E_t [\xi_T K] = \xi_t K e^{\lambda(\sigma_1 + \sigma_2)(T-t)}$.
Thus the expected yield can be rewritten as

\[ K e^{\lambda(\sigma_1 + \sigma_2)(T-t)} \frac{e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)}}{e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)}} [G_t - C_{BS}(G_t, K, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1)] \]

Since the elasticity of \( \xi_t \) with respect to \( V_t \) is bigger than one, the expected yield is an increasing function of \( V_t \). The expected yields on structured and non-structured securities move in opposite directions, which implies that structured securities should be viewed as hedge assets. This potentially explains the popularity of structured securities with investors and third-party issuers.

**D. Default Cost**

The prior analysis used a zero cost of default. This assumption is now relaxed to investigate the trade-off between the cost of default and the agency cost of asset substitution. For instance, consider the case in which the cost of default is proportional to firm value. Since default for the structured securities in Figure 2 occurs at high asset levels, the default cost may be much higher than for non-structured securities. Thus by structuring the debt to solve the asset substitution problem, the owner may be trading off a lower agency cost for a higher default cost.

Note that a non-zero default cost does not affect the ability of structured financing to solve the asset substitution problem. The functional form of the owner’s payoff (see Tables 2-5) remains valid. Thus the owner is still able to specify \( \{\theta_t : t \in [0, T]\} \) to induce the first best strategy. However, a non-zero default cost does impact the debt’s payoff. To raise \( V_0 - W \) at time 0, the debt’s face value typically will be higher than in the case without costly default. Thus the financing restrictions on \( K \) (for structured debt) and \( b \) (for structured convertible debt) must be altered to accommodate the time 0 value of the default cost.

Letting \( \delta \) denote the cost of default as a proportion of firm value, the default cost for structured debt is \( \delta V_T 1_{\{V_T < \xi_T \}} \), where \( 1_{\{\cdot\}} \) is the indicator function. Similarly, the default cost for non-structured debt is \( \delta V_T 1_{\{V_T < K\}} \). Replacing \( K \) in these expressions with \( K_1 \) gives the default cost for structured and non-structured convertible debt, respectively. The time \( t \) value of the default cost is calculated by taking an expectation with respect to the risk-adjusted measure \( Q \) and discounting at the rate \( r \). Closed-form expressions are given in the Appendix.
The top four graphs in Figure 3 illustrate the trade-off between the cost of default and the agency cost of asset substitution. The first best pair is \( \{\mu_2, \sigma_2\} \) and \( \sigma_1 > \sigma_2 \). The solid curves represent structured debt, which achieves the first best. The dashed curves represent non-structured debt in which the owner engages in asset substitution by shifting to the high volatility pair \( \{\mu_1, \sigma_1\} \) after the debt is in place. Moving clockwise from the top left, the top four graphs show the agency cost of substitution (which is zero for structured debt), the default probability, the total cost (agency cost plus default cost), and the default cost. Since the total cost curves intersect, structured debt may not always turn out to be the lower cost alternative. However, structured debt does allow the owner to eliminate the total cost at time \( T \) on the set of states for which the owner retains control of the firm. This is not true for non-structured debt since the owner still bears the cost of choosing a second best operating strategy, even if the firm does not default.

The bottom two graphs in Figure 3 illustrate the default costs and the conversion values for structured convertible debt (solid curves) and non-structured convertible debt (dashed curves). These two graphs use \( \mu_1 - \sigma_1 \lambda = \mu_2 - \sigma_2 \lambda \) and \( \sigma_2 > \sigma_1 \), which is consistent with Panels A and B of Table 4. Since the agency cost is zero for both types of debt, the default cost is also the total cost. Note that the total cost for structured debt may be considerably higher than that for non-structured debt. However, the set of states for which the total cost is incurred is different for the two types of securities. A similar statement can be made about the set of states for which the conversion option is valuable. This supports the earlier statement that structured securities are hedge assets, i.e., an investor can diversify by holding both types in the same portfolio.

**E. Other Characteristics**

To design a structured security that solves the asset substitution problem, the owner either must know the project parameters (see Tables 2-4) or must have knowledge of the average project volatility or the range of volatilities (see Section II.F). In these cases, the security’s design depends either directly or indirectly on parameters other than the first best. For example, suppose \( \{\mu_2, \sigma_2\} \) are the first best parameters and \( \{\mu_1, \sigma_1\} \) are the second best parameters. The first best firm value is \( V_t e^{(\mu_2-\sigma_2 \lambda - r)(T-t)} \), which depends only on the first best parameters. However, the structured debt in Panel A of Table 3 depends on \( \sigma_1 \), the second best volatility. As \( \sigma_1 \) increases, \( K \) must
increase due to the financing restriction. All else equal, the face value of the issued debt is higher as the second best project becomes more volatile, even though the owner never deviates from the first best. This shows how seemingly innocuous parameters of the investment opportunity set, in this case \( \sigma_1 \), become relevant when the firm issues a structured security. Relative to an all-equity firm, a firm that finances itself with structured debt faces additional exposure to shifts in the investment opportunity set. This suggests that structured financing is less suitable for companies that face unpredictable investment opportunities. On the other hand, firms in relatively stable industries are probably the best candidates for issuing structured debt. This latter statement appears to be consistent with the issuers in Table 1.

F. Structuring to Avoid Default

For structured debt with a time \( T \) face value of \( \xi_T K \), default may occur when the total assets are at a relatively high level. This leads to potentially high default costs, as shown in Figure 3. Since the elasticity of \( \xi_t \) with respect to \( V_t \) depends on \( \theta_t \), the question arises as to whether or not a debt contract can be structured to avoid default altogether. This section shows that such a structure is possible. The debt is structured to behave like straight debt when the firm is not in financial distress, but at the first sign of financial distress the debt acquires equity-like characteristics and continues to behave in this manner until its final maturity date.

Using the framework from Section II, suppose the firm issues structured debt whose time \( T \) payoff is \( \min [V_T, \xi_T K] \), where \( \xi_t \) is given by (6) and \( V_0 > K \). Suppose the owner chooses the first best pair \( \{\mu_2, \sigma_2\} \) and define the stopping time \( \tau = \inf \{t \geq 0 : V_t \leq K\} \). Thus \( \tau \) is the first time the firm’s assets fall to the level \( K \). Let \( \theta_t = 0 \) for all \( t \leq \tau \) and let \( \theta_t = \sigma_2 \) for all \( t > \tau \).

This type of structured security has the following properties. Prior to time \( \tau \), \( \xi_t = 1 \) since \( \theta_t = 0 \). In this case, the time \( t \) face value is \( K \) and the firm is not in financial distress since \( V_t > K \). If \( V_t \) never hits \( K \) prior to \( T \), the debtholders receive \( K \) at time \( T \). On the other hand, if \( \tau \) occurs before \( T \), the firm is in a state of potential financial distress since \( V_\tau = K \). But now \( \xi_t \) begins to vary since \( \theta_t = \sigma_2 \). Since \( \theta_t \) matches the volatility of \( V_t \), the elasticity of \( \xi_t \) with respect to \( V_t \) is now equal to 1. The assets \( V_t \) continue to grow at the rate \( \mu_2 - \sigma_2 \lambda > 0 \), but random fluctuations in \( V_t \) are matched one for one by fluctuations in \( \xi_t \). Thus \( V_T > \xi_T K \) at time \( T \) and
the firm never defaults on its debt.\footnote{This can also be seen by noting that the volatility of $G_t$ under the $R$ measure vanishes when $\theta_t = \sigma_2$ (see equation (11)). Since $G_T = K$ and $\mu_2 - \sigma_2 \lambda > 0$, $G_T > K$ at time $T$, which implies that $V_T \geq \xi_T K$.}

Since the firm never defaults, the value of this structured debt contract is

\[
E^Q_t \left[ e^{-r(T-t)} \min \{V_T, \xi_T K\} \right] = E^R_t \left[ \frac{\xi_T}{\xi_T} e^{-r(T-t)} \xi_T K \right] = \xi_t e^{-r(T-t)} K.
\]

At time 0, the value is $e^{-rT} K$. Even though the final payoff is $\xi_T K$, which is risky, the structured debt is valued at time 0 as if it is a riskless security. The reason for this is that the potential upside (resulting from a random increase in $V_t$ after time $\tau$) is balanced against the potential downside (resulting from a random decrease in $V_t$ after time $\tau$) such that the time 0 value coincides with that of riskless debt.\footnote{This debt structure is equivalent to having straight riskless debt with face value $K$ and a forward contract whose value is zero at time 0 and whose payoff is $\xi_T K - K$ at time $T$. Decomposing the debt structure in this manner shows why structured debt can be interpreted as a bundled hedge (Smithson and Chew (1992); Chidambaran, Fernando, and Spindt (2001)).}

The owner’s value at time $t$ is $\xi_t \left[ e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} G_t - e^{-r(T-t)} K \right]$, which shows that \( \frac{\partial^2 f}{\partial G^2} = 0 \). Thus (12) implies that the owner’s optimal choice is the first best pair \( \{\mu_2, \sigma_2\} \), which was assumed earlier. This example shows how structuring can simultaneously solve the asset substitution problem without incurring potentially high default costs. This is also a case in which the owner and the debtholders agree on the operating policy of the firm (see Section II.E).

V. Concluding Remarks & Future Research

This article shows how the asset substitution problem can be solved via capital structure design in a dynamic setting. While this article enhances our understanding of dynamic asset substitution, several open questions remain. First, it would be interesting to examine whether short term debt rebalancing, as discussed in Childs, Mauer, and Ott (2005) and Ju and Ou-Yang (2005), is a true substitute for structured financing. For example, do firms with similar characteristics use debt rebalancing and structured financing interchangeably? Leary and Roberts (2005) provide empirical evidence on dynamic rebalancing, but they do not control for structuring. While collateral pledges and covenants are sometimes used to mitigate asset substitution due to investment flexibility, the evidence in MacKay (2003) suggests that risk shifting due to production...
flexibility is harder to contract away. Thus it would be interesting to see if firms with high levels of production flexibility are more frequent users of structured debt.

Second, as Myers (1977) has argued, leveraged firms may underinvest in positive NPV projects since the benefit may go to the debtholders. While the empirical evidence supporting Myers’ (1977) argument is mixed (see Barclay and Smith (1995), Stohs and Mauer (1977), Titman and Wessels (1988)), this article suggests a new insight to the problem. Specifically, a firm should use structured debt whose face value varies with the moneyness of the firm’s growth options, thus mitigating the incentive of equityholders to pass up good projects. Along similar lines, it would be interesting to examine how the relationship between debt capacity and growth options, as discussed in Barclay, Morelec, and Smith (2006), is altered in the presence of structured financing.

Third, structuring appears to be a win-win outcome from the perspective of both the issuer and the buyer. The issuer is able to solve the asset substitution problem, which eliminates the agency cost that would otherwise be borne by the firm’s equityholders. On the other hand, the expected yield on structured debt, unlike that of non-structured debt, may be an increasing function of the firm’s assets. In an economy with multiple firms whose assets are cross-sectionally correlated, a bond investor can diversify his portfolio by adding structured securities. Structured securities are hedge assets and should command a high price in practice. It would be interesting to investigate empirically the diversification benefits from holding structured securities.

Lastly, the analysis reveals there are multiple capital structures that solve the asset substitution problem for each set of parameter restrictions (see Tables 2-5). Although certain practical features such as taxes have been omitted, the multiplicity of capital structures suggests that it should be relatively easy for firms to adopt a capital structure that eliminates the agency cost of dynamic asset substitution. This point of view is consistent with the survey evidence in Graham and Harvey (2001, Table 10). Furthermore, since there are multiple structures that solve the asset substitution problem, it seems unlikely that all of these would be rendered infeasible once the omitted features such as taxes are accommodated. It would be interesting to see if these statements are supported by financial managers in practice.
Appendix

The function $D(G_t, t)$: The function $D(G_t, t)$ is discussed in Section II.C. For the region $G_t \leq g(t)$ the function $D(G_t, t)$ is

$$D(G_t, t) = G_t N(d_1) + e^{-c(T-t)} [K_2 N(d_2) - K_1 N(d_3)]$$

$$+ \left( \frac{\sigma_{\text{max}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{G_t}{g(t)} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \sigma_{\text{max}} \sigma_{\text{max}} N(d_4) - N(d_5)$$

$$- \left( \frac{\sigma_{\text{max}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{L}{K_1} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \sigma_{\text{max}} N(d_6)$$

$$- \left( \frac{\sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{L}{K_2} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \sigma_{\text{min}} N(d_7),$$

while for the region $G_t > g(t)$ the function $D(G_t, t)$ is

$$D(G_t, t) = G_t N(-d_8) + e^{-c(T-t)} [K_2 N(d_9) - K_1 N(d_{10})]$$

$$+ \left( \frac{\sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{G_t}{g(t)} \right)^{\sigma_{\text{min}} - \sigma_{\text{min}}} \sigma_{\text{min}} N(d_{11}) - N(d_{12})$$

$$- \left( \frac{\sigma_{\text{max}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{L}{K_1} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \sigma_{\text{max}} N(d_{13})$$

$$- \left( \frac{\sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) G_t \left( \frac{L}{K_2} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \sigma_{\text{min}} N(d_{14}),$$

where $N(.)$ is the standard normal cumulative distribution function. The expressions for $d_1$-$d_{14}$ are

$$d_1 = \frac{\log \left( \frac{G_t}{K_1} \right) + c(T-t) + \frac{1}{2} \sigma_{\text{max}}^2 (T-t)}{\sigma_{\text{max}} \sqrt{T-t}},$$

$$d_2 = \frac{\log \left( \frac{G_t}{K_1} \right) - \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \log \left( \frac{K_2}{L} \right) + c(T-t) - \frac{1}{2} \sigma_{\text{max}}^2 (T-t)}{\sigma_{\text{max}} \sqrt{T-t}},$$

$$d_3 = \frac{\log \left( \frac{G_t}{K_1} \right) + c(T-t) - \frac{1}{2} \sigma_{\text{max}}^2 (T-t)}{\sigma_{\text{max}} \sqrt{T-t}},$$

$$d_4 = \frac{\log \left( \frac{G_t}{K_1} \right) + \log \left( \frac{K_2}{L} \right) + c(T-t) - \frac{1}{2} \sigma_{\text{max}}^2 (T-t) + \sigma_{\text{min}} \sigma_{\text{max}} (T-t)}{\sigma_{\text{max}} \sqrt{T-t}},$$

$$d_5 = \frac{\log \left( \frac{G_t}{K_1} \right) - \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \log \left( \frac{K_2}{L} \right) + c(T-t) - \frac{1}{2} \sigma_{\text{max}}^2 (T-t) + \sigma_{\text{min}} \sigma_{\text{max}} (T-t)}{\sigma_{\text{max}} \sqrt{T-t}},$$

$$d_6 = \frac{\log \left( \frac{G_t}{K_1} \right) + \log \left( \frac{K_2}{L} \right) + c(T-t) + \frac{1}{2} \sigma_{\text{max}}^2 (T-t)}{\sigma_{\text{max}} \sqrt{T-t}}.$$
\[ d_7 = \frac{\log \left( \frac{G_t}{L} \right) - \frac{\sigma_{\max}}{\sigma_{\min}} \log \left( \frac{K_2}{L} \right) + c(T - t) + \frac{1}{2} \sigma_{\max}^2 (T - t)}{\sigma_{\max} \sqrt{T - t}}, \]
\[ d_8 = \frac{\log \left( \frac{G_t}{K_2} \right) + c(T - t) + \frac{1}{2} \sigma_{\min}^2 (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_9 = \frac{\log \left( \frac{G_t}{K_2} \right) + c(T - t) - \frac{1}{2} \sigma_{\min}^2 (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_{10} = \frac{\log \left( \frac{G_t}{K_2} \right) - \frac{\sigma_{\min}}{\sigma_{\max}} \log \left( \frac{K_1}{L} \right) + c(T - t) - \frac{1}{2} \sigma_{\min}^2 (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_{11} = \frac{\log \left( \frac{L}{G_t} \right) - \log \left( \frac{K_2}{L} \right) - c(T - t) + \frac{1}{2} \sigma_{\min}^2 (T - t) - \sigma_{\min} \sigma_{\max} (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_{12} = \frac{\log \left( \frac{L}{G_t} \right) + \frac{\sigma_{\min}}{\sigma_{\max}} \log \left( \frac{K_1}{L} \right) - c(T - t) + \frac{1}{2} \sigma_{\min}^2 (T - t) - \sigma_{\min} \sigma_{\max} (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_{13} = \frac{\log \left( \frac{L}{G_t} \right) + \frac{\sigma_{\min}}{\sigma_{\max}} \log \left( \frac{K_1}{L} \right) - c(T - t) - \frac{1}{2} \sigma_{\min}^2 (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]
\[ d_{14} = \frac{\log \left( \frac{L}{G_t} \right) - \log \left( \frac{K_2}{L} \right) - c(T - t) - \frac{1}{2} \sigma_{\min}^2 (T - t)}{\sigma_{\min} \sqrt{T - t}}, \]

where \( \log(\cdot) \) denotes the natural logarithm. Equation (14) can be verified by direct substitution of (45)-(46) and its partial derivatives. It can also be verified that \( \frac{\partial D}{\partial G} \) and \( D \) are continuous at \( g(t) \). Letting \( K_1 = Le^{-b \sigma_{\max}} \) and \( K_2 = Le^{b \sigma_{\min}} \) for some \( b > 0 \), it is straightforward to verify that \( d_4 = d_5, d_6 = d_7, d_{11} = d_{12}, \) and \( d_{13} = d_{14} \), which simplifies (45)-(46). Furthermore, with \( K_1 = Le^{-b \sigma_{\max}} \) and \( K_2 = Le^{b \sigma_{\min}} \), the first and second derivatives of \( D(G_t, t) \) can be written as

\[
\frac{\partial D}{\partial G} = \begin{cases} 
N(d_1) - e^{-b(\sigma_{\max} - \sigma_{\min})}N(d_6), & G_t \leq g(t) \\
N(-d_8) - e^{-b(\sigma_{\max} - \sigma_{\min})}N(d_{13}), & G_t > g(t)
\end{cases}
\]

and

\[
\frac{\partial^2 D}{\partial G^2} = \begin{cases} 
\left( \frac{1}{G_t \sigma_{\max}\sqrt{2\pi(T-t)}} \right) e^{-\frac{1}{2}(d_1)^2} \left[ 1 - \left( \frac{G_t}{g(t)} \right)^{\frac{2b}{\sigma_{\max}(T-t)}} \right], & G_t \leq g(t) \\
\left( \frac{1}{G_t \sigma_{\min}\sqrt{2\pi(T-t)}} \right) e^{-\frac{1}{2}(d_8)^2} \left[ \left( \frac{g(t)}{G_t} \right)^{\frac{2b}{\sigma_{\min}(T-t)}} - 1 \right], & G_t > g(t)
\end{cases}
\]

The sign of \( \frac{\partial D}{\partial G} \) is always positive, while the sign of \( \frac{\partial^2 D}{\partial G^2} \) in (47) is determined by the term in square brackets. It is obvious that \( \frac{\partial^2 D}{\partial G^2} > 0 \) if \( G_t < g(t) \), \( \frac{\partial^2 D}{\partial G^2} < 0 \) if \( G_t > g(t) \), and \( \frac{\partial^2 D}{\partial G^2} = 0 \) if
\( G_t = g(t) \).

**Default cost:** Analytical expressions for the proportional default cost, which is discussed in Section IV.D, are presented here. Let \( \mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda \) so that \( \{\mu_2, \sigma_2\} \) is the first best pair. If \((\sigma_t - \theta_t)^2\) is constant, then Black-Scholes pricing applies. Letting \( \beta^2 \) denote the constant value of \((\sigma_t - \theta_t)^2\), the time \( t \) value of the default cost is

\[
E_t^Q \left[ e^{-r(T-t)} \delta V_T 1_{\{V_t < \xi_t K\}} \right] = \delta V_t e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} N \left( \frac{\log \left( \frac{\xi_t K}{V_t} \right) - (\mu_2 - \sigma_2 \lambda + \frac{1}{2} \beta^2)(T-t)}{\beta \sqrt{T-t}} \right),
\]

where \( \delta \) is the default cost as a proportion of time \( T \) firm value. For the structured debt (solid curve) in the middle left graph of Figure 3, \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \) and thus \( \beta = \sigma_1 \). For the non-structured debt (dashed curve) in the middle left graph of Figure 3, let \( \xi_t = 1 \), \( \beta = \sigma_1 \), and replace \( \mu_2 - \sigma_2 \lambda \) with \( \mu_1 - \sigma_1 \lambda \).

If \((\sigma_t - \theta_t)^2\) switches values, then Black-Scholes pricing no longer applies. Following Section II.C, let \( \sigma_{\text{max}} = \max[\sigma_1, \sigma_2] \), let \( \sigma_{\text{min}} = \min[\sigma_1, \sigma_2] \), and define \( g(t) \) as in (13). If the owner finances the firm with convertible debt, the time \( t \) value of the default cost is

\[
E_t^Q \left[ e^{-r(T-t)} \delta V_T 1_{\{V_t < \xi_t K_{13}\}} \right].
\]

For the region \( G_t \leq g(t) \) this value is equal to

\[
\delta \xi_t G_t e^{(c-r)(T-t)} \left\{ N(-d_1) + \left( \frac{\sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) \left( \frac{L}{K_1} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \frac{\sigma_{\text{min}} - \sigma_{\text{max}}}{\sigma_{\text{max}} - \sigma_{\text{min}}} N(d_6) \right\},
\]

while for the region \( G_t > g(t) \) this value is equal to

\[
\delta \xi_t G_t e^{(c-r)(T-t)} \left( \frac{\sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} \right) \left\{ \left( \frac{G_t}{g(t)} \right)^{\sigma_{\text{max}} - \sigma_{\text{min}}} \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{\sigma_{\text{min}}} N(d_{12}) + \left( \frac{L}{K_1} \right)^{\sigma_{\text{min}} - \sigma_{\text{max}}} \frac{\sigma_{\text{min}} - \sigma_{\text{max}}}{\sigma_{\text{max}} - \sigma_{\text{min}}} N(d_{13}) \right\},
\]

where \( d_4, d_6, d_{12}, \) and \( d_{13} \) are given above. For the bottom left graph of Figure 3, \( c = \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda \), \( \sigma_{\text{max}} = \sigma_2 \), and \( \sigma_{\text{min}} = \sigma_1 \). For the structured debt (solid curve) in this graph, \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \). For the non-structured debt (dashed curve) in this graph, \( \theta_t = 0 \) for all \( t \) and thus \( \xi_t = 1 \) and \( G_t = V_t \).

\[\blacksquare\]
Table 1
Sample of Indexed Note Offerings, 1980-2005

<table>
<thead>
<tr>
<th>Issuing Company</th>
<th>Year of Issue</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sunshine Mining</td>
<td>1980</td>
<td>Silver indexed notes&lt;sup&gt;1&lt;/sup&gt;</td>
</tr>
<tr>
<td>Oppenheimer &amp; Co.</td>
<td>1981</td>
<td>Indexed to NYSE trading volume&lt;sup&gt;2&lt;/sup&gt;</td>
</tr>
<tr>
<td>Marathon Oil</td>
<td>1986</td>
<td>Oil indexed notes&lt;sup&gt;3&lt;/sup&gt;</td>
</tr>
<tr>
<td>Pegasus Gold</td>
<td>1986</td>
<td>Gold indexed notes&lt;sup&gt;2&lt;/sup&gt;</td>
</tr>
<tr>
<td>Salomon Brothers</td>
<td>1986</td>
<td>Indexed to the S&amp;P 500&lt;sup&gt;2&lt;/sup&gt;</td>
</tr>
<tr>
<td>Merrill Lynch</td>
<td>1987</td>
<td>Indexed to the NYSE Composite&lt;sup&gt;2&lt;/sup&gt;</td>
</tr>
<tr>
<td>Eastman Kodak</td>
<td>1987</td>
<td>Gold indexed notes&lt;sup&gt;1&lt;/sup&gt;</td>
</tr>
<tr>
<td>Magma Copper</td>
<td>1988</td>
<td>Copper indexed notes&lt;sup&gt;4&lt;/sup&gt;</td>
</tr>
<tr>
<td>Presidio Oil &amp; Gas</td>
<td>1989</td>
<td>Natural gas indexed notes&lt;sup&gt;5&lt;/sup&gt;</td>
</tr>
<tr>
<td>Shin Etsu Chemical</td>
<td>1991</td>
<td>Inverse oil indexed notes&lt;sup&gt;2&lt;/sup&gt;</td>
</tr>
<tr>
<td>Freeport-McMoRan</td>
<td>1993</td>
<td>Gold indexed notes&lt;sup&gt;1&lt;/sup&gt;</td>
</tr>
<tr>
<td>Sallie Mae</td>
<td>1997</td>
<td>CPI indexed notes&lt;sup&gt;6&lt;/sup&gt;</td>
</tr>
<tr>
<td>Zurich Financial</td>
<td>2002</td>
<td>Indexed to the Swiss Market Index (SMI)&lt;sup&gt;7&lt;/sup&gt;</td>
</tr>
<tr>
<td>J.P Morgan Chase</td>
<td>2005</td>
<td>Indexed to the Russell 2000&lt;sup&gt;8&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

### Table 2
Feasible Capital Structures for $\mu_2 - \sigma_2 \lambda > \mu_1 - \sigma_1 \lambda$ and $\sigma_2 > \sigma_1$

#### A. Non-Structured Debt

<table>
<thead>
<tr>
<th>Description</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Owner’s payoff at $T$</td>
<td>$\max [0, V_T - K]$</td>
</tr>
<tr>
<td>Financier’s payoff at $T$</td>
<td>$\min [V_T, K]$</td>
</tr>
<tr>
<td>Structuring factor $\theta_t$</td>
<td>$0$</td>
</tr>
<tr>
<td>Optimal strategy at $t$</td>
<td>${\mu_2, \sigma_2}$</td>
</tr>
<tr>
<td>Owner’s value at $t_0$</td>
<td>$(\mu_2 - \sigma_2 \lambda - r)(T - t_0)$</td>
</tr>
<tr>
<td>Financing restriction on $K$</td>
<td>$V_0 - W = e^{(\mu_2 - \sigma_2 \lambda - r)T} [V_0 - BS(V_0, K, T - t, \mu_2 - \sigma_2 \lambda, \sigma_2)]$</td>
</tr>
</tbody>
</table>

| Parameter conditions | $K_1 = Le^{-b\sigma_2}$ and $K_2 = Le^{b\sigma_1}$ |
| Structuring factor $\theta_t$ | $0$ if $G_t \leq g(t)$, $\sigma_1 + \sigma_2$ if $G_t > g(t)$, $g(t) = Le^{-(\mu_2 - \sigma_2 \lambda)(T - t_0)}$ |
| Optimal strategy at $t$ | $\{\mu_2, \sigma_2\}$ |
| Owner’s value at $t_0$ | $\xi e^{(\mu_2 - \sigma_2 \lambda - r)T} D(G_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T - t, \mu_2 - \sigma_2 \lambda, \sigma_2, \sigma_1)$ |
| Local equity restriction on $L$ | $L = V_0 e^{(\mu_2 - \sigma_2 \lambda - r)T}\xi^{2\sigma_1(T - t_0)}$ |
| Financing restriction on $b$ | $V_0 - W = e^{(\mu_2 - \sigma_2 \lambda - r)T} [V_0 - D(V_0, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T, \mu_2 - \sigma_2 \lambda, \sigma_1)]$ |

Two different capital structures are illustrated for the case in which the pair $\{\mu_2, \sigma_2\}$ exhibits both a higher risk-adjusted return and a higher volatility relative to $\{\mu_1, \sigma_1\}$. Both capital structures are consistent with the first-best strategy, which is to choose $\{\mu_2, \sigma_2\}$ for all $t$. The restriction on $L$ (bottom panel) ensures that the owner’s claim is locally equity-like (a zero second derivative) at time $0$. The financing restrictions on $K$ (top panel) and $b$ (bottom panel) ensure that the issued security is fairly priced at time $0$. Thus the owner captures the first-best NPV and the agency cost of external financing is zero. $BS$ denotes the Black-Scholes formula, $D$ is given in the text, and $G_t = V_0 \xi^{t_0}^{-1}$. 

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Table 3
Feasible Capital Structures for $\mu_2, \sigma_2 > \mu_1, \sigma_1$ and $\sigma_2 < \sigma_1$

A. Structured Debt

<table>
<thead>
<tr>
<th>Parameter/Strategy</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Owner’s payoff at $T$</td>
<td>$\max [0, V_T - \xi_T K]$</td>
</tr>
<tr>
<td>Financier’s payoff at $T$</td>
<td>$\min [V_T, \xi_T K]$</td>
</tr>
<tr>
<td>Structuring factor $\theta_t$</td>
<td>$\sigma_1 + \sigma_2$</td>
</tr>
<tr>
<td>Optimal strategy at $t$</td>
<td>${\mu_2, \sigma_2}$</td>
</tr>
<tr>
<td>Owner’s value at $T$</td>
<td>$\xi_t e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} C_{BS}(G_t, K, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1)$</td>
</tr>
<tr>
<td>Financing restriction on $K$</td>
<td>$V_0 - W = e^{(\mu_2 - \sigma_2 \lambda - r)T} [V_0 - C_{BS}(V_0, K, T, \mu_2 - \sigma_2 \lambda, \sigma_1)]$</td>
</tr>
</tbody>
</table>

B. Structured Convertible Debt

<table>
<thead>
<tr>
<th>Parameter/Strategy</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Owner’s payoff at $T$</td>
<td>$\max [0, V_T - \xi_T K_1] - \max [0, V_T - \xi_T K_2]$</td>
</tr>
<tr>
<td>Financier’s payoff at $T$</td>
<td>$\min [V_T, \xi_T K_1] + \max [0, V_T - \xi_T K_2]$</td>
</tr>
<tr>
<td>Parameter conditions</td>
<td>$K_1 = L e^{-b \sigma_1}$ and $K_2 = L e^{b \sigma_2}$</td>
</tr>
<tr>
<td>Structuring factor $\theta_t$</td>
<td>$\sigma_1 + \sigma_2$ if $G_t \leq g(t)$, $\sigma_1$ if $G_t &gt; g(t)$</td>
</tr>
<tr>
<td>Optimal strategy at $t$</td>
<td>${\mu_2, \sigma_2}$</td>
</tr>
<tr>
<td>Owner’s value at $T$</td>
<td>$\xi_t e^{(\mu_2 - \sigma_2 \lambda - r)(T-t)} D(G_t, L e^{-b \sigma_1}, L e^{b \sigma_2}, L, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1, \sigma_2)$</td>
</tr>
<tr>
<td>Local equity restriction on $L$</td>
<td>$L = V_0 e^{(\mu_2 - \sigma_2 \lambda + (\sigma_1 - \sigma_2)^2/4)(T-t)}$</td>
</tr>
<tr>
<td>Financing restriction on $b$</td>
<td>$V_0 - W = e^{(\mu_2 - \sigma_2 \lambda - r)T} [V_0 - D(V_0, L e^{-b \sigma_1}, L e^{b \sigma_2}, L, T-t, \mu_2 - \sigma_2 \lambda, \sigma_1, \sigma_2)]$</td>
</tr>
</tbody>
</table>

Two different capital structures are illustrated for the case in which the pair $\{\mu_2, \sigma_2\}$ exhibits a higher risk-adjusted return but a lower volatility relative to $\{\mu_1, \sigma_1\}$. Both capital structures are consistent with the first-best strategy, which is to choose $\{\mu_2, \sigma_2\}$ for all $t$. The restriction on $L$ (bottom panel) ensures that the owner’s claim is locally equity-like (a zero second derivative) at time 0. The financing restrictions on $K$ (top panel) and $b$ (bottom panel) ensure that the issued security is fairly priced at time 0. Thus the owner captures the first-best NPV and the agency cost of external financing is zero. $C_{BS}$ denotes the Black-Scholes formula, $D$ is given in the text, and $G_t = V_0 
^{\mu_2 - \sigma_2 \lambda}$. 

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Four different convertible debt structures are illustrated for the case in which the pairs \(\{\mu_1, \sigma_1\}\) and \(\{\mu_2, \sigma_2\}\) exhibit the same risk-adjusted return but have different volatilities. For each structure, the owner’s payoff at \(T\) is \(\max\{0, V_T - \xi_T K_1\}\) and the financier’s payoff at \(T\) is \(\min\{0, V_T - \xi_T K_2\}\) and the local equity restriction on \(L\) is \(L_T = V_t e^{(\mu_2 - \sigma_2^2/2)T + \sigma_2^2/2}T\). The financing restriction on \(b\), which ensures that the issued security is fairly priced at time \(0\), is \(V_0 - W = e^{(\mu_2 - \sigma_2^2/2)T}V_0 - f_0\), where \(f_0 = f(V_0, 0)\) is the owner’s value at time \(0\). The four structures induce different strategies but all four deliver the first-best NPV (since \(\mu_2 - \sigma_2^2 = \mu_1 - \sigma_1^2\)). Thus the agency cost of external financing is zero. \(D\) is given in the text and \(G_t = V_t \xi_t^{-1}\).

### Table 4
Feasible Convertible Debt Structures for \(\mu_2 - \sigma_2^2 = \mu_1 - \sigma_1^2\) and \(\sigma_2 > \sigma_1\)

#### A. Non-Structured Convertible Debt

<table>
<thead>
<tr>
<th>Structuring factor (\theta)</th>
<th>(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal strategy at (t)</td>
<td>({\mu_2, \sigma_2}) if (V_t \leq g(t)), ({\mu_1, \sigma_1}) if (V_t &gt; g(t))</td>
</tr>
<tr>
<td>Owner’s value at (t)</td>
<td>(e^{(\mu_2 - \sigma_2^2/2)T}V_t - L e^{(\mu_2 - \sigma_2^2)T} - \frac{1}{2}\sigma^2(T-t))</td>
</tr>
</tbody>
</table>

#### B. Structured Convertible Debt

<table>
<thead>
<tr>
<th>Structuring factor (\theta)</th>
<th>(\sigma_1 + \sigma_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal strategy at (t)</td>
<td>({\mu_1, \sigma_1}) if (G_t \leq g(t)), ({\mu_2, \sigma_2}) if (G_t &gt; g(t))</td>
</tr>
<tr>
<td>Owner’s value at (t)</td>
<td>(e^{(\mu_2 - \sigma_2^2)T}V_t - L e^{(\mu_2 - \sigma_2^2)T} - \frac{1}{2}\sigma^2(T-t))</td>
</tr>
</tbody>
</table>

#### C. Structured Convertible Debt

<table>
<thead>
<tr>
<th>Structuring factor (\theta)</th>
<th>(\sigma_1 + \sigma_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal strategy at (t)</td>
<td>({\mu_1, \sigma_1}) if (G_t \leq g(t)), ({\mu_2, \sigma_2}) if (G_t &gt; g(t))</td>
</tr>
<tr>
<td>Owner’s value at (t)</td>
<td>(e^{(\mu_2 - \sigma_2^2)T}V_t - L e^{(\mu_2 - \sigma_2^2)T} - \frac{1}{2}\sigma^2(T-t))</td>
</tr>
</tbody>
</table>

#### D. Structured Convertible Debt

<table>
<thead>
<tr>
<th>Structuring factor (\theta)</th>
<th>(\sigma_1 + \sigma_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal strategy at (t)</td>
<td>({\mu_1, \sigma_1}) if (G_t \leq g(t)), ({\mu_2, \sigma_2}) if (G_t &gt; g(t))</td>
</tr>
<tr>
<td>Owner’s value at (t)</td>
<td>(e^{(\mu_2 - \sigma_2^2)T}V_t - L e^{(\mu_2 - \sigma_2^2)T} - \frac{1}{2}\sigma^2(T-t))</td>
</tr>
</tbody>
</table>
Table 5


A. Structured Convertible Debt with \( \nu^2 (k'(0.5I))^2 + 4\sigma^2 k(0.5I)k'(0.5I) < 0 \)

<table>
<thead>
<tr>
<th>Structuring factors at ( t )</th>
<th>( \theta_{1,t} )</th>
<th>Max and Min volatilities</th>
<th>Optimal strategy at ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \nu(0.5I) + \frac{\nu^2(k'(0.5I))^2}{2\nu k(0.5I)} )</td>
<td>( \sigma_{\text{min}} = -\frac{\nu^2(k'(0.5I))^2}{2\nu k(0.5I)} )</td>
<td>( I_{1,t}^* = I_{2,t}^* = 0.5I )</td>
</tr>
<tr>
<td>( G_t \leq g(t) ) and ( \theta_{2,t} = \nu k(0.5I) )</td>
<td>( G_t &gt; g(t) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Owner’s value at \( t \)

\[ \xi_t e^{(\mu-\sigma\lambda)(0.5I)-\nu/2} (\max \{ G_t - D(G_t, L^{\nu}e^{-b\nu}e^{-\nu^2/2}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\text{max}}, \sigma_{\text{min}} \}, 0) \]

Financing restriction on \( b \)

\[ I - W = e^{(\mu-\sigma\lambda)(0.5I)-\nu/2} (\max \{ G_t - D(G_t, L^{\nu}e^{-b\nu}e^{-\nu^2/2}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\text{max}}, \sigma_{\text{min}} \}, 0) \]

B. Structured Convertible Debt with \( \nu^2 (k'(0.5I))^2 + 4\sigma^2 k(0.5I)k'(0.5I) > 0 \)

<table>
<thead>
<tr>
<th>Structuring factors at ( t )</th>
<th>( \theta_{1,t} )</th>
<th>Max and Min volatilities</th>
<th>Optimal strategy at ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \nu(0.5I) + \frac{\nu^2(k'(0.5I))^2}{2\nu k(0.5I)} )</td>
<td>( \sigma_{\text{min}} = 2\nu k(0.5I) )</td>
<td>( I_{1,t}^* = I_{2,t}^* = 0.5I )</td>
</tr>
<tr>
<td>( G_t \leq g(t) ) and ( \theta_{2,t} = \nu k(0.5I) )</td>
<td>( G_t &gt; g(t) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Owner’s value at \( t \)

\[ \xi_t e^{(\mu-\sigma\lambda)(0.5I)-\nu/2} (\max \{ G_t - D(G_t, L^{\nu}e^{-b\nu}e^{-\nu^2/2}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\text{max}}, \sigma_{\text{min}} \}, 0) \]

Financing restriction on \( b \)

\[ I - W = e^{(\mu-\sigma\lambda)(0.5I)-\nu/2} (\max \{ G_t - D(G_t, L^{\nu}e^{-b\nu}e^{-\nu^2/2}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\text{max}}, \sigma_{\text{min}} \}, 0) \]

Two convertible debt structures are illustrated for the case in which a scale function \( k(\cdot) \) is used to aggregate the project returns. The restriction on \( k(\cdot) \) and its derivatives, which is given in the top row of each panel, ensures that the second-order condition is satisfied. The owner’s payoff at \( T \) is \( \max [0, X_T - \xi_T K_1] \max [0, X_T - \xi_T K_2] \) and the financier’s payoff at \( T \) is \( \min [X_T, \xi_T K_1] + \max [0, X_T - \xi_T K_2] \). Each capital structure induces the owner to undertake the first-best strategy of equal investment in the two projects for all \( t \). Thus, the agency cost of external financing is zero. For each panel \( K_1 = L^{\nu}e^{-b\nu}e^{-\nu^2/2}, K_2 = L^{\nu}e^{-b\nu}e^{-\nu^2/2}, \) and \( g(t) = L^{\nu}e^{-b\nu}e^{-\nu^2/2}[0, \xi_T K_1] \max [0, X_T - \xi_T K_2] \) where \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) are given in the table. The financing restriction on \( b \) ensures that the issued security is fairly priced at time 0, while the restriction \( L = G_t e^{(\mu-\sigma\lambda)(0.5I)-\nu/2} \min \{ \sigma_{\text{max}}, \sigma_{\text{min}} \} \) ensures that the owner’s claim is locally equity-like (a zero second derivative) at time 0. The function \( D \) is given in the text and \( G_t = \hat{X}_t^{\nu/2} \).
Figure 1
Agency Cost and Investment
The left graph shows the agency cost of asset substitution when the owner chooses a constant investment strategy that deviates from the first best. The agency cost is defined as the first best firm value divided by the second best firm value, minus one. The first best firm value is \( X_0 e^{(\mu-\lambda)k(0.5)T-\tau T} \) and the second best firm value is \( X_0 e^{(\mu-\lambda)(k(I_1,t)+k(I-I_1,t))T-\tau T} \). The scale function is \( k(z) = z^\gamma \), where \( \gamma \) equals 0.1 in the bottom curve (dotted), 0.3 in the middle curve (dashed), and 0.5 in the top curve (solid). The other parameters are \( \mu = 0.2 \), \( \sigma = 0.25 \), \( \lambda = 0.3 \), \( I = 100 \), and \( T = 1 \). The right graph uses equation (37) to illustrate how \( I_{1,t} \) varies with \( \theta_{1,t} \). The other parameters for this graph are \( \gamma = 0.5 \), \( \sigma = \nu = 0.25 \), and \( \theta_{2,t} = 0 \). 

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Debt Payoffs

The non-structured payoffs (top row) are \( \min [V_T, K] \) for the left graph and \( \min [V_T, K_1] + \max [0, V_T - K_2] \) for the right graph. The structured payoffs (bottom row) are \( \min [V_T, \xi_T K] \) for the left graph and \( \min [V_T, \xi_T K_1] + \max [0, V_T - \xi_T K_2] \) for the right graph. All graphs use \( V_0 = 100 \) and \( W = 50 \); thus the owner raises 50 via external debt financing. To raise 50, \( K = 93.36 \) in the top left graph, \( K = 52.60 \) in the bottom left graph, and \( K_1 = 52.56 \) and \( K_2 = 230.20 \) for both of the right-hand graphs. The structured securities use \( \theta_t = \sigma_1 + \sigma_2 \) for all \( t \). The first best pair is \( (\mu_2, \sigma_2) \). The lower left graph uses \( \sigma_1 = 0.30 \) and \( \sigma_2 = 0.25 \), which is consistent with Table 3. The other graphs use \( \sigma_1 = 0.25 \) and \( \sigma_2 = 0.30 \), which is consistent with Tables 2 and 4. Other parameters are \( \mu_2 - \sigma_2 \lambda = 0.125 \), \( r = 0.05 \), and \( T = 1 \).
Figure 3
Agency Cost, Default Cost, and Conversion Value
The top four graphs compare structured debt (solid curves) to non-structured debt (dashed). The structured debt achieves the first best; for the non-structured debt, the owner is assumed to engage in asset substitution by choosing the high volatility pair \( \{\mu_1, \sigma_1\} \) for all \( t \). The four graphs show the agency cost of asset substitution (which is zero for the first best), the default probability, the default cost, and the total cost (agency plus default). The bottom two graphs compare structured (solid) and non-structured (dashed) convertible debt. Both achieve the first best. The left (right) graph shows the default cost (conversion option value). All graphs use \( V_0 = 100 \) and \( W = 50 \). To raise \( V_0 - W = 50 \), the structured debt uses \( K = 52.64 \), the non-structured debt uses \( K = 92.78 \), and both convertible securities use \( K_1 = 52.60 \) and \( K_2 = 230.08 \). All of the structured securities use \( \theta_1 = \sigma_1 + \sigma_2 \) for all \( t \). Other parameters are \( r = 0.05, T-t = 0.5, \) and \( \delta = 0.1 \); the top four graphs use \( \mu_2 - \sigma_2 \lambda = 0.125, \mu_1 - \sigma_1 \lambda = 0.05, \sigma_1 = 0.30, \) and \( \sigma_2 = 0.25 \); the bottom two graphs use \( \mu_2 - \sigma_2 \lambda = \mu_1 - \sigma_1 \lambda = 0.125, \sigma_2 = 0.30, \) and \( \sigma_1 = 0.25 \).
References


