Uncertainty Risk and Cross-sectional Returns: Theory and Evidence

(Job Market Paper)

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ABSTRACT

In this paper we develop a continuous-time general equilibrium model in a representative exchange economy with incomplete information. We show, in a multiple assets setting, that state uncertainty risk is priced and commands additional (state-dependent) premium. It is affected by both the investor’s estimate of the state of the economy, as well as the uncertainty about her estimation. Moreover, noisier stocks/firms respond more strongly to uncertainty shocks, which may help rationalize Fama-French size and book-to-market factors. We formally test our model using the ASA-NBER survey of professional forecasters, GDP growth rate data, and Fama-French size and book-to-market portfolios. The empirical evidence is largely consistent with the model’s predictions. The estimated uncertainty risk premium is both statistically and economically significant (over 4% per year), and the Fama-French factors lose explanatory power for the cross-sectional returns after controlling for the uncertainty risk.
I. Introduction

Classical asset pricing models of Sharpe-Lintner-Black (1964,1965,1972), Lucas (1978), Merton (1971, 1973) and Cox, Ingersoll, and Ross (1985) have set up a systematic structure for modern research on the financial market. Unfortunately, a series of empirical results provide challenges to those models, in terms of both time series and cross-sectional behavior of asset returns. One important assumption in the classic models is that the informational structure is fully observable. This is a very strong assumption. In the real world, investors cannot observe the true state of the growth rate of the economy, and they have to infer it from observed yet noisy information (e.g., dividends, asset returns, news event, etc.). We refer to this source of risk as state uncertainty risk. This paper provides a general equilibrium framework in which the state uncertainty risk is priced, and the cross-section of stock returns is partially explained by the uncertainty risk factors.

The motivation of this paper is two-fold. On the theoretical front, a number of papers have studied the effect of incomplete information on stock returns, volatility, and interest rate at the market level (See Detemple (1986), Gennotte (1986), Dothan and Feldman (1986) for early development, and Veronesi (1999,2000) and Yan (2001) for recent treatment). Wang (1993), and He and Wang (1995), among others, model asymmetric information and examine its impact on asset return and volatility. Detemple and Murthy (1994), Basak (1998), Basak and Croitoru (2000), and Zapatero (1998) directly model the agents’ differences in beliefs. There is also a growing behavioral literature trying to explain asset pricing anomalies as results of investors’ irrationalities. (See Hirshleifer (2001) for a comprehensive survey and references therein).

These papers mainly focus on the dynamics of a single (or market) security; the behavior of the cross-sectional returns in the presence of incomplete information has rarely been examined. Intuitively, aversion to state uncertainty may lead to a demand for additional risk premium. In CAPM or beta pricing framework, if the stocks are correlated with the investor’s updating process, the state uncertainty may serve as an additional factor besides market return. This issue deserves a closer study.
On the empirical front, several papers examine the effect of learning on investors’ portfolio choice in a partial equilibrium setting. Kandel and Stambaugh (1996) are among the first to address the importance of parameter uncertainty on Bayesian investors’ portfolio choice problem. More recent work in this framework include Barberis (2000), Xia (2001) and Handa and Tiwari (2001). Because of the partial equilibrium nature, these papers specify the state variables or factors governing the return process as exogenous, without justifying for the underlying uncertainty risk in the economy. To better understand the effect of the fundamental uncertainty in the economy on stock market returns, we need a formal test of the cross-sectional returns and the fundamental uncertainty in a general equilibrium setting.

In this paper, we build a continuous time equilibrium model for the cross-sectional stock returns, which serves as a theoretical background for empirical testing. The economy is a standard Lucas-type exchange economy with $n$ real output (or dividend) processes, except that the true dividend growth rate follows an unobservable Markov-chain process. For simplicity, we specify two states for the dividend growth rate (i.e., the good state and the bad state), which interchange randomly according to the continuous-time Markov chain. There are $n$ stocks and a risk-free bond market. The representative investor cannot observe the true state, so she updates her estimate of the state of the economy based on observed realizations of the dividends and asset prices. She then decides her consumption and portfolio choice given her information set; and the stock returns and interest rate are endogenously determined upon imposing equilibrium conditions.

This paper shares a similar information structure as in Veronesi (2000), but the issues addressed here are very different. First, Veronesi (2000) models a single asset and examines the time series properties of the market return, while our model is in a multiple assets setting and we focus on the cross-sectional behavior of the asset returns in the presence of the state uncertainty. Second, Veronesi (2000) investigates the impact of the precision of an extra noisy signal on the equity premium, in the context of the consumption CAPM. In contrast, the present paper examines how the state uncertainty affects cross-sectional asset returns in a more general intertemporal CAPM framework. Our analysis leads to an intuitive result: aversion to uncertainty risk commands additional premium, and stocks’ different sensitivities to
uncertainty shocks partially explains the observed return differences. Third, our model helps to rationalize Fama and French (1993) factors as proxies for the uncertainty risk. Finally, we conduct various tests of our model and the empirical evidence is largely supportive of the model’s predictions. Our main results can be summarized as follows:

1. Even in the presence of state uncertainty, the stock returns (and the market return) can still be written as a function of their covariances with the aggregate consumption. More importantly, for the purpose of this paper, we show that the cross-sectional returns can be expressed as the weighted sum of their covariances with the market return and their covariances with the estimate of the uncertainty of the economy. This is consistent with our intuition: in the presence of incomplete information, the state uncertainty introduces another dimension of priced risk. Moreover, the premium on the uncertainty risk is increasing in both the investor’s estimate of the current state of the economy, as well as the uncertainty about the estimate. Intuitively, when the investor’s estimate of the current state of the economy is low, higher uncertainty implies a better chance for the economy to revert to the good state; while when the estimate is high, any fluctuation on the estimate means a possible deterioration of the economy, which may lead the investor to demand more premium. Given the estimate of the economy, the less certain about the estimate (or higher state uncertainty), the higher premium the investor demands because of her aversion to state uncertainty.

2. Secondly, for reasonable range of parameters, we show that noisier firms (firms with more volatile dividend growth rate) have higher covariance with uncertainty shocks, and have higher premia. Empirically, this may rationalize the Fama and French (1993) size and book-to-market factors to some extent. Intuitively, smaller firms tend to be noisier and more vulnerable to the changes of economic conditions, hence may respond more strongly to the state uncertainty. As to the book-to-market effect, Lakonishok, Shleifer and Vishny (1994) argue that value firms are not fundamentally riskier. However, Zhang (2002) and Petkova and Zhang (2002) provide evidence that value stocks are riskier in bad times. In our context, their results may indicate that value firms have higher covariance with uncertainty risk at economic downturns, hence they command higher premium. Since the Fama-French factors SMB and HML essentially capture the return differences among firms with different size and growth level, and our model
implies that these differences can be explained by their different requirements for the uncertainty risk premium, these factors may well be proxies for the uncertainty risk factors. As we discuss below, the empirical evidence largely supports this argument.

3. We also conduct a series of tests and the empirical results generally support our model. We propose two measures to proxy the uncertainty factor. The first is a direct measure using the ASA-NBER survey of professional forecasters, obtained from Federal Reserve Bank of Philadelphia. The second is an indirect measure by using the fitted value of a Probit model where the dependent variable takes a value of one whenever the GDP growth rate is greater than its unconditional median, and zero otherwise. The explanatory variables are the instruments that have been used to forecast GDP in the literature. We construct a number of variables to proxy for different aspects of uncertainty. We use the Fama-French 25 size and book-to-market portfolios as the testing portfolios. To conduct the tests, we first derive the conditional model’s implications in an unconditional setting. Specifically, the unconditional expected returns are determined by the market beta, the beta on the estimate of the economy, and the beta on the uncertainty about the estimate. We then run two kinds of tests. The first is based on the two-pass cross-sectional regressions and the second is based on the GMM estimate of the equivalent pricing kernel. The results suggest that uncertainty factors are priced in both sets of tests. In the two-pass cross-sectional regressions, the estimates of uncertainty factor premia are significantly different from zero with or without the inclusion of the Fama-French factors. The size and book-to-market factors lack marginal explanatory power when the uncertainty factors are included, which is consistent with our model’s prediction. For the equivalent pricing kernel models, the impact of Fama-French factors is significantly reduced in the presence of the uncertainty factors, and we cannot reject the null hypothesis that the loadings on SMB and HML are jointly zero when all uncertainty factors are included. But we can strongly reject the opposite hypothesis that the loadings on the uncertainty factors are all zero.

4. To evaluate the economic significance of the uncertainty risk premium, we group all common stocks traded at NYSE/AMEX/NASDAQ into ten portfolios based on their uncertainty betas. The average returns display an increasing pattern from low uncertainty beta portfolios to high uncertainty beta portfolios. The return difference among portfolio 10 (high uncertainty
beta) and portfolio 1 (low uncertainty beta) is 6 percent per year and is statistically significant. The differences cannot be explained by the CAPM or by a four-factor model that includes the three Fama-French factors and the momentum factor. Moreover, we construct an uncertainty factor mimicking portfolio as the return differences among high and low uncertainty beta portfolios. We then repeat the two-pass cross-sectional regression procedure. The estimated uncertainty factor premium is 4.4 percent per year and significant at 5 percent level. After controlling for this mimicking portfolio, none of the estimated premia for SMB, HML, and the momentum factor is statistically significantly different from zero at the 10 percent level.

Several recent papers also try to explore the ability of incomplete/asymmetric information in explaining the asset return “anomalies”. Lewellen and Shanken (2002) show how learning can generate ex-post return predictability. Kogan and Wang (2002) and Liu, Pan, and Wang (2002) introduce model uncertainty. David and Veronesi (2002) examine the empirical relationship between inflation and earnings uncertainty and asset volatility. Goetzmann and Massa (2001) and Ghysels and Juergens (2001) empirically test models of asymmetric information and/or different beliefs. Our paper is perhaps more related to Massa and Simonov (2002), who also investigate whether learning about uncertainty is a priced factor. However, there are several important differences between the two papers. We develop a continuous time version of the Lucas type model, in which the investors face uncertainty about state variables, while the Massa and Simonov (2002) discrete trading model ignores consumption and partially depends on changing priors. Empirically we examine the effect of fundamental uncertainty on the cross-sectional returns, while they test the impact of “induced” uncertainty, which is proxied by measures of pricing errors. Furthermore, their paper investigates asymmetric information as well, while ours focuses on fundamental uncertainty by assuming a representative agent.

The rest of the paper is organized as follows. The next section discusses the model and the investor’s updating process. Section III provides the solutions and detailed analysis. Section IV discusses the econometric specifications. Section V conducts empirical tests and analyzes the results. Section VI provides concluding remarks.
II. The Model

A. The Economy and Information Structure

We assume an exchange economy with a representative agent. There are \( n \) exogenous dividend processes and hence an aggregate dividend process. We further assume that the first \( n-1 \) dividend processes and the aggregate dividend follow:

\[
\begin{align*}
    dD &= I_D ((A + B \theta_t) dt + \Sigma dZ_t), \\
    &\text{where } I_D \text{ is a } n \times n \text{ diagonal matrix with the first } n-1 \text{ elements as } D_i, \text{ and the } n-\text{th element as the aggregate dividend, } D_A. \text{ A and } B \text{ are both } n \times 1 \text{ vectors and } Z_t \text{ is a } n\text{-vector Brownian motion. }
\end{align*}
\]

Note that here we model the first \( n-1 \) dividend processes and the aggregate dividend, so that the last dividend \( D_n \) is defined as \( D_n \equiv D_A - \sum_{i=1}^{n-1} D_i \). We assume all the regularity conditions apply and all the dividend processes are well behaved.

The variable \( \theta_t \) is an unobservable state variable, which takes two values \((\underline{\theta}, \overline{\theta})\). The \( \theta_t \) may continuously switch values according to the Markov infinitesimal matrix:

\[
G \equiv \begin{pmatrix}
    -pf & pf \\
    pf & p(1-f) \\
    p(1-f) & -p(1-f)
\end{pmatrix},
\]

where \( f \) is the unconditional probability, \( Pr(\theta = \overline{\theta}) \).

The information structure discussed here is similar to that in Veronesi (2000), who studies the dynamics of a single asset in a multiple states setting. Moreover, he adds a noisy signal and examines the impact of the precision of that signal on equity premium in the context of consumption CAPM. Our purpose here is to study the property of cross-sectional returns in the presence of imperfect information, hence, we model the economy in a multiple assets setting. Since the unobservable state variable itself can generate insightful result, we ignore the external signal to keep the model simple. Also, we obtain our solution to the model in the more general intertemporal CAPM framework, and discuss how the state uncertainty affects asset returns as an additional risk factor besides the market factor.
There are $n$ stocks, $S$, with each stock representing the claim of an individual dividend. The share of each stock is normalized to 1. There is also a bond market with an instantaneous interest rate $r_t$. The total share of the bond is zero. Both the stock return dynamics and the interest rate are determined endogenously in equilibrium.

The representative investor has a standard power utility. She chooses her consumption rate $c_t$, and the vector of portfolio weights $\alpha_t$, and solves

$$\max_{c_t, \alpha_t} E \left( \int_t^\infty e^{-r(s-t)} \frac{c_t^{1-\gamma}}{1-\gamma} ds \right),$$

subject to the budget constraint on her wealth $W_t$:

$$dW_t = W_t \left[ \alpha_t' IS(dS + Ddt) + (1 - \alpha_t' 1_n)r dt \right] - c_t dt,$$

where $1_n$ is a vector of $n$ ones.

Note that since the state variable $\theta$ is not observable, the investor’s problem is not Markovian.

### B. Updating Process

Since the investor cannot observe the true state of the economy, she has to infer it from the observed information. Specifically, the investor updates the posterior probability $\pi_t \equiv Pr(\theta_t = \theta | F_t)$ based on realized dividends. The updating process is given by (See Lipster and Shiryaev (2001), chapter 9):

$$d\pi_t = p(f - \pi_t) dt + \pi_t \left[ (A + B(\bar{\theta} - \mu_\theta))' \Sigma^{-1} d\tilde{Z} \right],$$

where $\mu_\theta = E(\theta | F_t) = \pi_t \bar{\theta} + (1 - \pi_t) \underline{\theta}$, and $d\tilde{Z} = \Sigma^{-1} \left[ I_D^{-1} dD - (A + B \mu_\theta) dt \right]. \pi_t$ reflects the investor’s estimate about the state of the economy. When $\pi_t$ is high, the economy is more likely to be in the good state. When $\pi_t$ is low, the economy is more likely to be in the bad state. When $\pi_t$ is 0.5, the investor has least knowledge about the true state of the economy. So
the dynamics of $\pi_t$ also measure the changes of the uncertainty level.

Given the dynamics of $\pi_t$, the dividend processes can be rewritten as:

$$I_D^{-1}dD = (A + B\mu_0)dt + \Sigma d\tilde{Z}.$$  \hspace{1cm} (4)

It is now well known that the economy represented by equations (3) and (4) is informationally equivalent to the original economy, yet the new economy is fully observable and the dividend processes and the “new” state variable $\pi_t$ are jointly Markovian (see, for example, Detemple (1986), Gennotte (1986), and Dothan and Feldman (1986) for the case of Gaussian state variable; and David (1997) and Veronesi (1999) for Markov chain process). We can now solve for the equilibrium under this fully observable economy.

III. Cross-sectional Returns, Market Return, and State Uncertainty

A. The Market Equilibrium

Given the representative agent and the informationally equivalent economy, the market is effectively complete. The first order condition of the investor’s problem implies a state price density process

$$\xi_t = e^{-\rho t} \frac{u_c(D_A(t))}{u_c(D_A(0))} = e^{-\rho t} \left( \frac{D_A(t)}{D_A(0)} \right)^{-\gamma},$$  \hspace{1cm} (5)

where the aggregate dividend $D_A$ equals the total consumption in equilibrium. Applying Ito’s lemma to equation (5), and we have

$$\frac{d\xi_t}{\xi_t} = \left( -\rho - X(t)\mu_A(t) + \frac{1}{2}X(t)Y(t)\|\sigma_A\|^2 \right) dt + X(t)\sigma_A d\tilde{Z},$$  \hspace{1cm} (6)

where $\mu_A(t)$ is the expected growth rate of the aggregate dividend, $\sigma_A$ is the instantaneous volatility of the aggregate dividend (a $n$ dimensional vector), and

$$X(t) = -\frac{D_A u_{es}(D_A(t))}{u_c(D_A(t))} = \gamma$$

$$Y(t) = \frac{D_A u_{ss}(D_A(t))}{u_{cc}(D_A(t))} = \gamma + 1.$$  

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Given the state price density, any asset price can be expressed as

\[ S_i(t) = E_t \left[ \int_t^\infty \frac{\xi(s)}{\xi(t)} D_i(s) \, ds \right], \tag{7} \]

which also follows a diffusion process. Or in the differential form, the first order condition implies that for any asset, we have

\[ 1 = E_t \left[ \left( 1 + \frac{d\xi_t}{\xi_t} \right) \left( 1 + \frac{dS_{i,t}}{S_{i,t}} \right) \right]. \]

\[ \Rightarrow \]

\[ E_t \frac{dS_{i,t}}{S_{i,t}} + E_t \frac{d\xi_t}{\xi_t} = -E_t \left[ \frac{d\xi_t}{\xi_t} \frac{dS_{i,t}}{S_{i,t}} \right]. \tag{8} \]

When the asset is the risk-free bond, we have \( \frac{dS_{i,t}}{S_{i,t}} = r(t) dt \), and (8) implies that

\[ r(t) = -E_t \frac{d\xi_t}{\xi_t}. \tag{9} \]

Comparing equations (8), (9), and (6), and denote in the vector form the drift term of \( \frac{dS_t}{S_t} \) as \( \mu(t) \) and its instantaneous volatility matrix as \( \Omega_t \), we have

\[ r(t) = \rho + \gamma (A_A + B_A \mu_0) - \frac{1}{2} \gamma (\gamma + 1) \| \sigma_A \|^2, \tag{10} \]

where \( A_A \) and \( B_A \) are the elements of the drift term coefficients that are associated with the aggregate dividend. Further, the cross-sectional expected returns are given by:

\[ \mu(t) + \frac{D(t)}{P(t)} - r(t) = \gamma \Omega_t \sigma_A = \gamma \text{cov}_t \left( \frac{dS_t}{S_t}, \frac{dD_{A,t}}{D_{A,t}} \right) \tag{11} \]

Equation (11) shows that even in the presence of state uncertainty, the cross-sectional returns can still be written in terms of the covariances with the aggregate consumption. This follows from the fact that the whole economy and the investor’s updating processes are solely driven by the realization of the dividend processes; there is no other fundamental risk in this economy. This result links asset returns with consumption in the consumption CAPM frame-
work. In this paper we want to examine the direct impact of the state uncertainty as a risk factor on the cross-sectional returns, besides the indirect effect through the market factor. We address this issue in the next section and deliver our solution to the model in a more general intertemporal CAPM framework.

B. The Dynamics of the Cross-sectional Returns

As we discuss above, we are mainly interested in examining the impact of the state uncertainty on the cross-sectional returns. To link the asset returns with the state uncertainty risk, we next decompose the covariance term in equation (11). Note that the price for the market portfolio can be expressed as:

$$S_M(t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{n_r(D_A(s))}{n_r(D_A(t))} D_A(s) \, ds \mid F_t \right]$$

⇒

$$\frac{S_M(t)}{D_A(t)} = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \frac{D_A(s)}{D_A(t)} \right)^{1-\gamma} \, ds \mid F_t \right]$$

(12)

Veronesi (2000, Appendix A) provides the solution for a single asset in a one dimensional setting. It is straightforward to extend the solution to the multi-dimensional case, and the price of the market portfolio is given by:

$$S_M(t) = D_A(t) [\pi t C_1 + (1 - \pi t) C_2]$$

(13)

where $C_i$s are constants that are independent from $D_A$ and time, and can be viewed as the normalized price the investor is willing to pay relative to the dividend, at the good and bad state, if the investor could actually observe the true state. They are given by:

$$C = M^{-1}I_2,$$

(14)

where

$$M = -G - (1 - \gamma) \left[ \begin{array}{c} A_A + B_A \bar{\theta} \\ A_A + B_A \bar{\theta} \end{array} \right] + \left( \rho + \frac{1}{2} \gamma (1 - \gamma) \| \sigma_A \|^2 \right) I_2.$$
Applying Ito’s lemma to (13), we have,

$$\frac{dS_M(t)}{S_M(t)} = \frac{dD_A}{D_A} + \frac{(C_1 - C_2)}{C_1\pi_t + C_2(1 - \pi_t)} d\pi_t + \frac{(C_1 - C_2)}{C_1\pi_t + C_2(1 - \pi_t)} (\sigma_A'\sigma_\pi) dt,$$

(16)

where $\sigma_\pi$ is the instantaneous volatility vector of $d\pi$ as shown in equation (1).

From (16), the instantaneous volatility vector of market return is

$$\sigma_M = \sigma_A + \frac{(C_1 - C_2)}{C_1\pi_t + C_2(1 - \pi_t)} \sigma_\pi.$$

(17)

Equation (17) shows that the market return volatility can be expressed as the sum of the volatility of the aggregate dividend (and consumption) and the volatility of the updating process with a time-varying loading. Rearranging (17), we have

$$\sigma_A = \sigma_M + \frac{(C_2 - C_1)}{C_1\pi_t + C_2(1 - \pi_t)} \sigma_\pi.$$

(18)

Substituting equation (18) into (11), we have

$$\mu(t) + \frac{D(t)}{P(t)} - r(t) = \gamma \text{cov}_t\left(\frac{dS}{S}, \frac{dS_M}{S_M}\right) + \frac{C_2 - C_1}{C_1\pi_t + C_2(1 - \pi_t)} \text{cov}_t\left(\frac{dS}{S}, d\pi\right).$$

(19)

We can rewrite equation (19) as a more familiar expected return-beta form,

$$\mu(t) + \frac{D(t)}{P(t)} - r(t) = \lambda_{m,t}\beta_{m,t} + \lambda_{\pi,t}\beta_{\pi,t},$$

(20)

where $\lambda_{m,t} = \gamma \sigma^2_{m,t}$ and $\lambda_{\pi,t} = \gamma \frac{C_2 - C_1}{C_1\pi_t + C_2(1 - \pi_t)} \sigma^2_{\pi,t}$; $\beta_{m,t} \equiv \frac{\text{cov}_t(\frac{dS}{S}, \frac{dS_M}{S_M})}{\sigma^2_{m,t}}$ and $\beta_{\pi,t} \equiv \frac{\text{cov}_t(\frac{dS}{S}, d\pi)}{\sigma^2_{\pi,t}}$.

But $\sigma_{\pi,t} = [A\pi_t + B(\theta - \tilde{\theta})\pi_t(1 - \pi_t)]' \Sigma^{-1}$. And from equation (18), $\sigma^2_{m,t}$ is also a function of $\sigma^2_{\pi,t}$. So the risk premia are functions of both $\pi_t$ and $\pi_t(1 - \pi_t)$.

Equations (19) and (20) show that the cross-sectional returns are not only determined by the individual stocks’ covariances with the market return, as in the standard CAPM, but also determined by their covariances with the estimation of the state of the underlying economy. When $C_1 = C_2$, i.e., when the investor doesn’t care which state she is in and is willing to
pay the same in each state, then the second term vanishes. Alternatively, if the return is independent of the estimation uncertainty, the second term drops off too. But in general, the state uncertainty risk serves as an additional factor explaining the cross-sectional returns besides the market factor.

Veronesi (2000) shows that for the market portfolio, $C_1 < C_2$ whenever $\gamma > 1$. This implies that the premium on the uncertainty factor is positive. The higher $\beta_{\pi,t}$, or the higher the covariance between the return and the innovation on the posterior probability, the higher the required expected return. This is consistent with investors’ aversion to state uncertainty. Moreover, from equation (20), $\lambda_{\pi,t}$ is an increasing function of both $\pi_t$ and $\pi_t (1 - \pi_t)$. Intuitively, when $\pi_t$ is low, the economy is more likely to be in the bad state. The higher uncertainty implies a better chance for the economy to revert to the good state; while when $\pi_t$ is about 1, any fluctuation on $\pi_t$ means a deterioration of the economy, which may lead the investor to demand higher premium. Given $\pi_t$, the higher $\pi_t (1 - \pi_t)$, the investor is less certain about her estimate of the economy, and hence demand higher premium. Overall, equations (19) and (20) suggest that the uncertainty risk affects the cross-sectional returns directly through the covariance term, in addition to the indirect effect through the market return.

The covariance between the individual return and state uncertainty can be further analyzed. Note that the pricing equation (12) also applies to individual stocks. Define $\eta_i(t) = \left( \frac{D_i(t)}{D_i(0)} \right)^{-\gamma} \left( \frac{D_A(t)}{D_A(0)} \right)^{1-\gamma}$, and the individual stock prices can be expressed as:

$$\frac{S_i(t)}{D_i(t)} = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{\eta_i(s)}{\eta_i(t)}^{1-\gamma} ds | F_t \right].$$

(21)

Applying Ito’s lemma and conditional on $\theta$, we have

$$\frac{d\eta_i}{\eta_i} = \frac{1}{1-\gamma} \left( (A_i - \gamma A_A) + (B_i - \gamma B_A) \theta + \frac{1}{2} ||\sigma_A||^2 - \gamma \left( \sigma_A^T \sigma_i \right) \right) dt + \frac{1}{1-\gamma} (\sigma_i - \gamma \sigma_A) dZ.$$  

(22)

Mathematically equation (21) and (22) have the same format as equation (12) and (1), so we can apply the same method used for the market portfolio to individual stocks. The
individual price is given by:

\[ S_i(t) = D_i(t) [\pi t C_{1i} + (1 - \pi_t) C_{2i}], \text{ for } i = 1, n - 1, \]  

(23)

where \( C_{1i} \) and \( C_{2i} \) are independent of \( D_i \) and \( \theta_i \) but are functions of both \( \sigma_A \) and \( \sigma_i \).

Applying Ito’s lemma to (23), we have the dynamics of the individual return process. In particular, the instantaneous volatility vector is:

\[ V\left( \frac{dS_i}{S_i} \right) = \sigma_{D_i} + \frac{C_{1i} - C_{2i}}{C_{1i}\pi_t + C_{2i}(1 - \pi_t)} \sigma_{\pi}, \]  

(24)

and

\[ \text{cov}_t \left( \frac{dS_i}{S_i}, d\pi_t \right) = \left( \sigma_{D_i} + \frac{C_{1i} - C_{2i}}{C_{1i}\pi_t + C_{2i}(1 - \pi_t)} \sigma_{\pi} \right) \sigma_{\pi}'. \]  

(25)

The sign of \( \frac{C_{1i} - C_{2i}}{C_{1i}\pi_t + C_{2i}(1 - \pi_t)} \) is ambiguous since Cs are functions of \( \sigma_A \) and \( \sigma_{D_i} \). But if we assume the parameters are such that \( \frac{C_{1i} - C_{2i}}{C_{1i}\pi_t + C_{2i}(1 - \pi_t)} \) is insensitive to \( \sigma_{D_i} \), then equation (25) implies that noisier firms (in terms of the volatility of the dividend process) have higher covariance with uncertainty shocks. This suggests that noisier firms will command higher premium for the uncertainty risk. Empirically, this may rationalize the Fama and French (1993) size and book-to-market factors to some extent. Intuitively, smaller firms tend to be noisier and more vulnerable to the changes in economic conditions, hence they may respond more strongly to state uncertainty. As to the book-to-market effect, Lakonishok, Shleifer and Vishny (1994) argue that value firms are not fundamentally riskier. However, Zhang (2002) and Petkova and Zhang (2002) provide evidence that value stocks are riskier in bad times. In our context, their results may indicate that value firms have higher covariance with uncertainty risk at economic downturns, hence they command higher premium. Since the Fama-French factors SMB and HML essentially capture the return differences among firms with different size and growth level, our model implies that these differences may be explained by their different sensitivities to the uncertainty shocks. Vassalou (2002) documents that Fama-French factors may be proxies for forecasts of conditional GDP growth. Here we provide some fundamental reason why that would be the case. We empirically test the model’s implications in the next section.
IV. Econometric Specifications

Our model is in a continuous-time setting. To conduct empirical tests, we first need discretization. The discrete time version of equation (20) can be written as:

\[ E_t(r_{i,t+1}) = \lambda_{m,t} \beta_{im,t} + \lambda_{\pi,t} \beta_{i,\pi,t}, \]  

where \( \beta_{im,t} \equiv \text{cov}(r_{i,t+1}, r_{m,t+1}) / \sigma_{m,t}^2 \), and \( \beta_{i,\pi,t} \equiv \text{cov}(r_{i,t+1}, \pi_{t+1}) / \sigma_{\pi,t}^2 \); \( r_{i,t+1} \) and \( r_{m,t+1} \) are next period excess return for portfolio \( i \) and the market portfolio, respectively; and \( \lambda_{m,t} \) and \( \lambda_{\pi,t} \) are functions of both \( \pi_t \) and \( \pi_t (1 - \pi_t) \), as indicated in equation (20).

This is a conditional pricing model, while we need to test the model empirically using unconditional data. The following lemma shows that under the assumptions in Appendix A, the unconditional expected returns can be expressed as a three-factor model.

**Lemma 1**: Define \( \beta_{im} \equiv \text{cov}(r_{i,t+1}, r_{m,t+1}) / \sigma_{m,t}^2 \), and \( \beta_{i,\pi} \equiv \text{cov}(r_{i,t+1}, \pi_{t+1}) / \sigma_{\pi,t}^2 \). Under the assumptions in Appendix A, equation (26) implies that there are some constants \( (q_0, q_1, q_2, q_3) \), such that the unconditional expected returns can be expressed as

\[ E[r_{i,t}] = q_0 + q_1 \beta_{i,m} + q_2 \beta_{i,\pi} + q_3 \beta_{i,\pi(1-\pi)} \]  

**Proof**: See Appendix A.

Taking unconditional expectation in equation (27) is not new for testing conditional models. Jagannathan and Wang (1996) derive the conditional CAPM’s unconditional implications to test the conditional model. Shapiro (2002) uses a similar technique when testing the conditional investor recognition hypothesis (IRH) model. Given equation (27), we can apply standard techniques to estimate and test the model. We can then also examine whether the explanatory power of variables such as size and book-to-market diminishes as our model predicts.

We estimate and test equation (27) directly using the standard two-pass cross-sectional regressions. At the first pass, we regress individual returns on the factors to determine the \( \beta \)s for these factors; then at the second pass, we run the cross-sectional regression to determine
whether these factors are priced. As it turns out, the uncertainty factors are indeed priced, so we can further test whether Fama-French factors (HML, SMB) have marginal effect on cross-sectional returns when state uncertainty risk is already priced. As we discussed earlier, return differences HML and SMB can be explained by the different requirements for the uncertainty risk premium and these factors in turn serve as proxies for the uncertainty risk factors. Hence we expect that HML and SMB are less significant in the presence of uncertainty factors. Liew and Vassalou (2000) and Vassalou (2002) argue that Fama-French factors are proxies for the economic growth; the issue to be tested here is how they are related to the investor’s confidence about the economic growth.

Cochrane (2001) emphasizes the equivalent pricing kernel representation for the expected return-β models. In our model, equation (20) implies a stochastic discount factor (or pricing kernel) exists such that

\[ E_t(m_{t+1}r_{i,t+1}) = 0, \]

(28)

where \( r_{i,t+1} \)s denote the excess returns in the payoff space. For CAPM, \( m_{t+1} = 1 + br_{m,t+1} \). If uncertainty risk is priced, we would expect the pricing kernel to be:

\[ m_{t+1} = 1 + b_t r_{m,t+1} + c_t^{\text{uncertainty factors}} \]

(29)

We estimate models (28) and (29) using GMM. We test whether uncertainty factors are priced, and whether Fama-French factors have marginal explanatory power after controlling for the uncertainty factors.

V. Empirical Evidence

Equation (19) predicts that the cross-section of returns can be explained by their covariances with (a) the market return and (b) the uncertainty factor. In this section, we test the model’s implications empirically. To conduct empirical tests, we need to obtain proxies for the uncertainty factors. We propose alternative proxies in this paper. Given the proxies for the uncertainty factor, we run a series of tests proposed in Section III to analyze the empirical
implications of the state uncertainty risk.

**A. Data and Proxies for $\pi_t$**

In this sub-section, we provide two approaches to proxy the uncertainty factor $\pi_t$. The first one is to use some direct measure of the investor’s confidence level about the economy. In this paper we use the ASA-NBER survey of professional forecasters obtained from Federal Reserve Bank of Philadelphia. This dataset contains quarterly updates of these forecasters’ forecasts on a series of economic variables, such as GDP growth, CPI, interest rates, etc. The sample spans the period from the first quarter of 1969 to the last quarter of 2001. The variable we use as a proxy for the uncertainty risk factor is the forecast of the probability of recession for the current quarter, and for the following one to five quarters. One minus this probability would be a natural measure of the probability that the economy is in the good state. Based on this variable, we define the following measures to capture the different aspects of the fundamental uncertainty.

1. **PR\_MEAN**, defined as one minus the mean of the forecasts of probability of recession for the next quarter. This is a direct measure of the state variable $\pi_t$. Table I below reports two other measures PR\_MEAN3QT, which is the average of the mean forecast for the current and next 2 quarters, and PR\_MEDIAN, defined as one minus the median of the forecasts of the probability of recession for next quarter. They are very similar to PR\_MEAN.

2. **PR\_UNC**, the measure of uncertainty, defined as $\text{PR\_MEAN}(1-\text{PR\_MEAN})$. From the analysis in the previous sections, PR\_MEAN measures the investor’s estimate of the state of the economy, PR\_UNC can capture the uncertainty about this estimate. The higher the PR\_UNC, the higher the investor is uncertain about the current state. To see this, note that PR\_UNC peaks at $\text{PR\_MEAN} = .5$, at which level the investor’s estimate of the economy in the good state is the same as that in the bad state, hence the investor has the least knowledge about the true state of the economy. Thus PR\_UNC serves as a proxy for $\pi_t(1 - \pi_t)$. 

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3. PR_ERR, measured by the cross-sectional standard deviation of individual forecasts of the probability of recession. This variable captures the estimation error of $\pi_t$, which also reflects the uncertainty about the estimate of the state, and hence serves as another proxy for $\pi_t(1 - \pi_t)$.

There has been an increasing trend to use the survey data in the uncertainty literature (See e.g., Massa and Simonov (2002), and David and Veronesi (2002)). This is an ex-ante measure of investors’ estimate of the fundamental economy, which is appropriate for testing our model. Moreover, the variable we use, namely, the probability of recession for next quarter is a direct probability measure, which just suits the notion of $\pi_t$ in our two-state model. As a first pass check of the accuracy of these forecasts, we did a frequency test (not reported here). Out of 113 actual GDP growing quarters, 110 have a mean forecast greater than 50%. The forecasters seem to do a poorer job in forecasting recessions, with the accuracy rate around 50%. But since there are only 19 recession quarters, the overall forecasts in our sample period don’t seem to be biased. As a robustness check, we use means, medians, and the average of three quarters forecasts in our later tests, and the results are very similar.

Table I reports the descriptive statistics of the different uncertainty measures constructed from the ASA-NBER survey data. The average number of forecasters for each quarter is 37. The average forecast of the probability of a growing economy next quarter is 80%. This is different from PR_UNC, whose average value is 0.1316, a little more than half of its maximum value 0.25. This suggests that the two measures do capture the different aspects of the uncertainty risk. The average standard deviations among the forecasters are about 0.1328. Panel B compares three measures for $\pi_t$, PR_MEAN, PR_MEAN3QT, and PR_MEDIAN. As we noted earlier, they are very similar to each other.

Another approach to proxy $\pi_t$ is to use GDP growth data and variables that have been shown to have forecasting power for GDP growth in the literature. Specifically, we set up a Probit model, where the dependent variable equals one whenever the GDP growth rate is above
its unconditional median, and it equals zero when it is below or equal to its median. Following
the literature (see, for example, Kandel and Stambaugh (1990)), we choose the following set of
explanatory variables: one quarter lagged GDP growth rate, one quarter lagged dividend yield
on S&P 500 index, the term spread defined as the difference of the quarterly rate between
10 year government bond and 1 year government bond, the default spread defined as the
difference of the quarterly rate between Moody’s BAA bond and AAA bond, and the 3-month
T-bill rate. The dividend yield and 3-month T-bill rate data are from Logical Information
Machines (LIM), and all the other data are downloaded from the website of the Federal
Reserve Bank of St. Louis. The sample spans from the first quarter of 1969 to the second
quarter of 1999. Given these variables, we then estimate the model. The fitted value is the
estimated probability that the economy is in good state. We therefore use it as our second
proxy for πt.

We use the Fama and French (1993) 25 size and book-to-market portfolios as testing port-
folios. They, as well as the market excess returns, the size (SMB) and book-to-market (HML)
factors and 1-month T-bill rate are all downloaded from Kenneth French’s website. We con-
struct the quarterly portfolio returns by compounding the monthly returns.

B. Two-pass Cross-sectional Regressions

We use the two-pass cross-sectional regression procedure to test the explanatory power
of the uncertainty factors and the other factors used in the literature. In the first step, we
estimate the loadings (βs) for each factor using the full sample. We then specify the following
cross-sectional equations for the second step:

\[ R_{t+1} = \gamma_{t+1} \beta, \]

where \( R_{t+1} \) is a \( n \times 1 \) vector of excess returns on \( n \) portfolios. In this paper, we use quarterly ex-
cess returns of Fama-French 25 size and book-to-market portfolios from 1969:01 to 2001:04,
altogether 132 quarters. \( \beta = [\beta_m, \beta_{PR\_MEAN}, \beta_{PR\_UNC}, \beta_{PR\_ERR}, \beta_{PR\_SMB}, \beta_{PR\_HML}]' \) is
a vector of factor loadings (except for the constant) from the first pass univariate regressions
of returns on the quarterly market excess return \( r_m; \) PR\_MEAN; PR\_UNC; PR\_ERR; SMB, defined as quarterly return difference between Fama-French small size portfolio and large size portfolio; and HML, defined as the quarterly return difference between Fama-French high book-to-market portfolio and low book-to-market portfolio. Since we have multiple portfolios, we estimate the \( \beta \)s using SUR (seemingly unrelated regression).

We consider two scenarios. In the first case, we estimate only the premia for the market factor and the uncertainty factors; in the second case, we add the Fama-French factors as explanatory variables. Both OLS and GLS estimates are reported. The weighting matrix in the GLS estimates is obtained from the sample residual covariance matrix of the time series regression of returns on the corresponding factors.\(^7\)

If uncertainty factors were priced, we would expect the loadings for the uncertainty factors to be statistically significant. Moreover, if, as the model predicts, the nosier firms have higher covariance with uncertainty factors, we would expect the Fama-French factors (SMB, HML) to have less explanatory power when we include uncertainty factors.

We conduct the cross-sectional regression analysis for both sets of proxies for the uncertainty factors. First, we present the results where the uncertainty factors are measured by the ASA-NBER survey data. As we discussed earlier, in the first pass, we estimate the \( \beta \) for each factor using the full sample. Then at the second pass, we run the cross-sectional regression using the estimated \( \beta \)s as explanatory variables for each quarter. We then test whether these factors are priced by examining the time series of the factor loadings. Table II reports the results for the second-pass cross-sectional regressions. The test reported below are consistent with our intuition: when controlling for uncertainty betas, the Fama-French betas are no longer significant.

We run the tests using mean, median, as well as the average of the current and next two quarters forecasts. (In the latter two cases, PR\_MEAN stands for PR\_MEDIAN and PR\_MEAN3QT, respectively, and PR\_UNC is defined accordingly.) The results are very sim-
ilar. To save space, we henceforth report the results using the average of 3 quarters’ forecasts from now on. We estimate two models in Table II. For Model 1 we only include the market excess return and the uncertainty factors. For Model 2 we include the Fama-French factors (SMB, HML) as well. The variable $\bar{\gamma}$ denotes the time series average of the factor loadings. If a factor is indeed priced, the average loading for that factor should be statistically different from zero. From Table II, we can see that the OLS estimates are basically consistent with GLS estimates. For GLS estimates, All the three variables PR_MEAN, PR_UNC and PR_ERR are priced in both Model 1 and Model 2. Among them, the premia for PR_MEAN and PR_UNC are positive. Although the coefficient for PR_ERR is negative, the magnitude is much smaller.8 The market premium is not significant, which is consistent with Fama and French (1992). Interestingly, with the inclusion of uncertainty factors, SMB and HML no longer have explanatory power cross-sectionally. This is consistent with our model’s prediction. Note that Fama-French factors SMB and HML basically capture the return differences among firms with different size and growth level. Our model implies that firms with different size and book-to-market levels may have different sensitivities to uncertainty shocks, and the size and book-to-market effects may actually be proxies for the uncertainty risk. Hence, in the presence of the uncertainty risk factors, their effects may no longer be significant. The results in Table II provide empirical support to this argument.9

To make sure that our results are not driven by particular choice of proxies for the uncertainty risk factor, we repeat our test using an alternative specification. Our second approach to proxy the uncertainty risk is to estimate a Probit model:

$$\text{Prob}(\text{state} = 1|x) = \phi (\beta^\prime x)$$  \hspace{1cm} (31)

where $\phi (\cdot)$ represents the cumulative distribution function for the standard normal distribution. Here we set state = 1 whenever the observed GDP growth rate is greater than the unconditional median growth rate. The vector $x$ contains the explanatory variables. As discussed in the previous subsection, the explanatory variables include one quarter lagged GDP growth rate, one quarter lagged dividend yield for S&P 500 index, the term spread defined as the difference
of the quarterly rate between 10 year government bond and 1 year government bond, the
default spread defined as the difference of the quarterly rate between Moody’s BAA bond and
AAA bond, and the 3-month T-bill rate. All rates are annualized.

Table III reports some of the descriptive statistics for the dependent and explanatory vari-
ables. The mean GDP growth rate from 1969 to 1999 is about 3.04% with a standard deviation
of 3.61%. The first order auto-correlation is 0.21. The average dividend yield over the sample
period is 3.65%. Consistent with the literature, the dividend yield is very persistent, with the
first order autocorrelation of 0.937, and it still has a autocorrelation of 0.577 even at the 10\textsuperscript{th}
order. The standard deviation is about 1.17%, suggesting that dividend yield is less volatile
than the GDP growth. The term spread, default spread and the 3-month T-bill are also quite
persistent. But the unit-root tests suggest that all the series are stationary.

Panel A of Table IV reports the estimation results for the Probit model expressed in equa-
tion (31). The estimated coefficients for lagged growth rate, term spread and 3-month T-bill
rate are significant at the 5% level. The coefficient for dividend yield is significant at the 10%
level. The negative sign is consistent with Kandel and Stambaugh (1990) in the context of
consumption growth. These results are generally consistent with the literature and reassure
that these variables do have some explanatory power over the future GDP growth.

Using the estimates in Panel A of Table IV, we calculate a fitted value of the variable $\overline{\text{state}}$ for
every quarter in the sample period. To be consistent with earlier measures of uncertainty,
we define two similar measures $PR\_MEAN \equiv \overline{\text{state}}$, and $PR\_UNC \equiv \overline{\text{state}} \cdot (1 - \overline{\text{state}})$ as the
two measures of the uncertainty factors for each quarter. The descriptive statistics are reported
in Panel B of Table IV. Note that the proxies for uncertainty risk based on the Probit model do
not necessarily possess the same properties as those based on the ASA-NBER survey data. In
the former case, the two states are defined based on whether the GDP growth in a quarter is
above or below the median growth rate, while in the latter case, the two states are defined as either growth or recession. The two scenarios do not always coincide, and hence the descriptive statistics in Panel B of Table IV differ from those in Table I.

We repeat the cross-sectional tests of equation (27) using the Fama-French 25 size and book-to-market portfolio. The results are reported in Table V.

As can be seen, the results in Table V are quite similar to those in Table II. As in Table II, we consider two scenarios. Model 1 contains only the market excess return and the uncertainty factors. In Model 2 we also add SMB and HML factors. Again, the two uncertainty factors are priced in both Model 1 and Model 2. The coefficient on PR\_MEAN is significant for both OLS and GLS estimates, and the coefficient on PR\_UNC is significant for GLS estimate. From the results of Model 2, the marginal effects of size and book-to-market are insignificant for both OLS and GLS estimates. In summary, the results in Table II and Table V confirm that the uncertainty risk is indeed priced. Moreover, in the presence of the uncertainty factors, the Fama-French factors lose power in explaining the cross-sectional returns. These findings are consistent with our conjecture that Fama-French factors may be proxies for the uncertainty risk since the return differences captured by those factors may reflect the different responses to uncertainty shocks implied by our model.

C. Pricing Kernel Estimates

As we discussed above, another approach to test our uncertainty model is by estimating the pricing kernel. We specify the pricing kernel for the most comprehensive model as:

\[ m_{t+1} = 1 + \beta' f_{t+1}, \]

(32)

where \( f_{t+1} \) includes the market factor, the demeaned uncertainty factors PR\_MEAN, PR\_UNC, and PR\_ERR, as well as the Fama-French factors SMB and HML.

We use GMM to estimate the pricing kernel in equations (28) and (29). Under this setup,
standard asset pricing theory tells us that the coefficients of the factors are proportional to the risk premia. We use the quarterly excess return of the same 25 size and book-to-market portfolios used in the cross-sectional tests as return data. Since the pricing kernel should be able to price any payoff in the payoff space, these 25 portfolios provide at least 25 moment conditions. The pricing kernel has at most 8 parameters to be estimated, so the system is over-identified.

If the model is correctly specified, we’d expect the uncertainty factors to be significant; moreover, if the size and book-to-market factors are indeed proxies for the uncertainty factors, we’d expect that they have less explanatory power when uncertainty factors are included.

Table VI reports the estimates of the pricing kernel. We consider four scenarios: Model 1 contains only three FF factors, Model 2 contains only the market factor and PR\_MEAN, Model 3 adds the market factor and FF factors SMB and HML, and Model 4 contains all factors. For the three-factor model, Fama and French (1993) estimate and report the results for individual portfolios. Here we examine the model’s pricing kernel implications. We note that all the estimates are significant at the 5% level. The positive sign on SMB does not necessarily mean a discount because of the covariance structure of the factors. The J-statistic (for overidentified restrictions) does not reject the model at the 5% level. This result confirms that when used alone, the Fama-French factors are priced.

In Model 2 where only the market factor and the demeaned PR\_MEAN are included, both factors are statistically significant at the 1% level. The negative signs are consistent with earlier cross-sectional regressions results and imply a positive premium on the uncertainty risk. The J-statistic decreases to 17.87, compared to 20.22 in Model 1. The associated p-value is 0.765, so that Model 2 is not rejected. This suggests that our model does a good job explaining at least the cross-sectional variation of the Fama-French 25 portfolios.

If, as we argued, Fama-French factors are proxies for the uncertainty risk, then we should observe that the explanatory power of the Fama-French factors weaken with the inclusion of
the uncertainty factor. In Model 3, the coefficient for PR_MEAN is still significant. Compared with model 2, the point estimate of PR_MEAN only slightly changes from -9.42 to -9.48. After controlling for PR_MEAN, although the loading for SMB is still significant, HML is no longer significant at the 5% level. The Wald test of null hypothesis that the loadings on SMB and HML are jointly zero are rejected, but the value of the $\chi^2$ statistic decreases drastically from 100.68 to 19.72. This suggests that the inclusion of the uncertainty risk factor indeed weakens the explanatory power of Fama-French factors. In Model 4 where all the factors are included, the coefficients of PR_UNC and PR_ERR are significant at the 1% level, suggesting that investors do care about the uncertainty about the estimate of the economy. PR_MEAN is not significant, which may be due to the interaction among the uncertainty factors. However, the null hypothesis that the loadings of all three uncertainty factors are zero is strongly rejected at any reasonable significant level, suggesting that the uncertainty risk is indeed priced. Moreover, now neither of the loadings on SMB or HML is significant at the 5% level. And we cannot reject the hypothesis that they are jointly zero at the 10% level. The results of Model 3 and Model 4 suggest although the investor’s estimate about the state of the economy is important, the Fama-French factors are not just business cycle related, they do proxy for the uncertainty about the estimate of the economy, as our model implies. Overall, the results in Table VI are largely consistent with the model’s predictions and reinforces our argument that Fama-French factors may be proxies for the uncertainty premium.

D. Economic Significance

In the previous sections we show that the uncertainty factor is a priced factor. Since the direct proxy we use is not a traded portfolio, the economic significance of the premium on the state uncertainty risk is not immediately evident. To evaluate the economic significance, we first form portfolios based on their sensitivities to uncertainty risk, and then construct a factor mimicking portfolio in a similar way to Fama-French factors SMB and HML. To make a general statement about the economic significance of the uncertainty risk premium in terms of returns, below we estimate a static version of the models (19) and (20), by treating the risk premia as constants. Thus, the estimated uncertainty premium can be think of as a mixture of
the effects of \( \pi_t \) and \( \pi_t(1 - \pi_t) \).

At the end of each year, we regress the monthly returns of all the common stocks (CRSP share code 10 and 11) listed on NYSE, AMEX, and NASDAQ on CRSP value-weighted index return and the PR_MEAN using the previous five years’ data. We then group the stocks into ten portfolios based on their PR_MEAN beta. We then record the monthly equally-weighted returns of these portfolios for a year. We repeat this procedure by updating uncertainty betas and rebalancing portfolios at the end of each year. This creates a monthly return series of each of the ten portfolios, covering the period from January 1973 to December 2001.

The first row of Table VII reports the monthly average returns of the uncertainty beta sorted portfolios in excess of the one-month treasury bill rate. Portfolio 1 has the lowest uncertainty beta while portfolio 10 has the highest. There is a clear pattern that portfolios with low uncertainty betas have lower returns than those with high uncertainty betas. The monthly excess returns increase almost monotonically from portfolio 1 (0.89 percent) to portfolio 10 (1.39 percent). The average monthly excess return difference between portfolio 10 and portfolio 1 is 0.5 percent per month, or 6 percent per year. The \( p \)-value of the null hypothesis that the average return difference between portfolio 10 and 1 is zero is 0.0098. So the return differences are both statistically and economically significant. These results are consistent with the model’s prediction that the stocks that are more sensitive to uncertainty risk will command higher risk premium.

Next we examine whether the return differences among uncertainty beta sorted portfolios can be explained by factor pricing models. If a portfolio return can be fully explained by a factor model, there shouldn’t be any systematic pricing errors so that the ”alphas” are insignificantly different from zero. If the uncertainty premia part can be explained by the factor models, ”alphas” across uncertainty beta sorted portfolios should no longer display any systematic patterns after adjusting for these factors. Table VII reports two commonly used factor models: Model 1 is the CAPM, and Model 2 is a four-factor model that includes the three
Fama-French factors plus the momentum factor. The data on the monthly factor returns are again downloaded from Kenneth French’s website. It is clear that CAPM cannot fully explain the returns on uncertainty beta sorted portfolios. On the one hand, the monthly returns for these returns decrease substantially after controlling for the market risk. For example, the estimated alpha for portfolio 1 is 0.2 percent per month, while that for portfolio 10 is 0.7 percent, both of which are much smaller than the simple average returns. On the other hand, except for portfolio 1, all the other estimated alphas are significantly different from zero. Moreover, even after controlling for the market factor, one can still observe an increasing trend on the estimated alphas. The alphas increase rapidly from low uncertainty beta portfolios to high uncertainty beta portfolios. The difference between the alphas for portfolio 1 and 10 is about 6 percent per year, again economically significant. The \( p \)-value of the null hypothesis that alpha_1 equals alpha_10 is 0.0121. The \( p \)-value for the GRS (Gibbons, Ross and Shanken (1989)) test of the null hypothesis that all alphas are jointly zero is 0.0017, so that we can reject the null at the 5 percent level.

The estimated alphas in the four-factor model display similar patterns to those in the CAPM. Most of the estimated alphas are significantly different from zero. One can also observe an increasing trend among the alphas across portfolios, although the alphas increase less rapidly than they in the CAPM. The difference between the alphas in portfolio 1 and 10 is about 4 percent per year, suggesting a significant amount of ”abnormal” return difference that are not accounted for by the four factors. The \( p \)-value of the null that alpha_1 equals alpha_10 is 0.0357, and that for the GRS test of the null that all 10 alphas are equal to zero is 0.0009, so that both hypotheses are rejected at 5 percent level. In summary, the results in Table VII confirm our model’s prediction that stocks with different sensitivities to uncertainty risks will have different returns, and the return differences cannot be explained by the market factor, Fama-French factors SMB, HML or the momentum factor.

Given the uncertainty beta sorted portfolios, we can construct an uncertainty risk factor mimicking portfolio. Specifically, we define \( UNC \equiv 1/3((portfolio1 + portfolio2 + portfolio3) - (portfolio8 + portfolio9 + portfolio10)) \). UNC reflects the compensation for uncertainty risk in terms of portfolio returns. Since UNC is the return on a managed portfolio we
can directly evaluate the economic significance of the uncertainty risk through the estimated premium on factor UNC. To do this, we repeat the Fama-Macbeth two-pass cross-sectional regressions. In the first pass, we estimate the multivariate betas in the first pass using last three years’ data. We update the betas each month in a rolling three-year window. In the second pass, we regress the 25 portfolio excess returns on the estimated betas each month. The time series averages of the coefficients \( \bar{\lambda} \) on betas are estimates of the risk premia for the factors. We consider 3 models. Model 1 contains the market factor and the uncertainty factor. Model 2 adds Fama-French factors SMB and HML, and Model 3 adds a momentum factor. Since this time we run multivariate regressions in the first pass, we can apply Shanken (1992)’s correction for the EIV problem. Panel A of Table VIII reports the mean estimates, \( p \)-value as well as the EIV corrected \( p \)-values.

In Panel A of Table VIII, for Model 1 where we have only the market factor and the uncertainty factor mimicking portfolio, the estimated risk premium for the uncertainty factor mimicking portfolio is about 0.46 percent per month, or 5.5 percent annually. The associated \( p \)-value is about 0.028. In Model 2, after controlling for SMB, HML, and UMD, the uncertainty premia decrease a little, but still at 4.4 percent per year, respectively. And the estimates are statistically significant at 5 percent level in both models. Strikingly, with the inclusion of the factor mimicking portfolio UNC, none of the estimates of the other factors are statistically significantly different from zero.\(^{10}\) This result is once again consistent with our argument that the Fama-French factors may proxy for the uncertainty risk. Note that the estimates of the constant terms are significant. This does not necessarily represent the pricing error though. As we explained earlier, we are estimating a static version of the model. The constant term may reflect the covariance of the time varying premia and betas, which are accounted for in the earlier estimates.

The above estimated premium on the uncertainty factor mimicking portfolio reflects a mixture of the effects of the estimate of the state uncertainty and the uncertainty about the
estimation. To separate the two, we construct two zero investment portfolios: one by longing and shorting portfolios sorted by PR_MEAN beta, one by PR_UNC beta. Then we repeat the above procedure and estimate the premia on each of the two portfolios. Panel B of Table VIII reports the estimation results. Again when we include only the market factor and uncertainty factor mimicking portfolios, the premia on both portfolios are positive and statistically significant at 5% level. The premium on PR_MEAN is 0.61% per month, while that on PR_UNC is a little smaller at 0.55% per month. When we add SMB, HML, and the momentum factor, the estimated premia decrease a little at 0.43% and 0.31% per month respectively, yet still statistically significant at 5% level. Consistent with the results in Panel A, neither of the estimated premia on SMB, HML, and UMD is statistically significant.

In summary, all of our empirical tests confirm that the uncertainty risk is indeed priced and has substantial economic significance. Furthermore, in the presence of the uncertainty factors, the Fama-French factors lose explanatory power for the cross-sectional returns, which is broadly consistent with our model’s predictions.

VI. Summary and Conclusions

In this paper, we developed a continuous-time general equilibrium asset-pricing model with incomplete information. The representative investor updates her estimate of the unobservable state variable (expected dividend growth rate) based on observed realizations of the dividends and asset returns. She then decides her consumption and portfolio choice given her information set; and the stock returns and interest rate are endogenously determined upon imposing equilibrium conditions. By constructing an informationally equivalent economy, we are able to maintain the Markovian property of the problem and solve for the interest rate and the cross-section of asset returns in equilibrium.

We find that even in the presence of estimation uncertainty, the expected returns can be written as a function of their covariances with the aggregate consumption/dividends. More importantly, the cross-sectional returns can be expressed as the weighted sum of their covariances with the market return and their covariances with the estimate of the uncertainty of the
economy. Intuitively, in the presence of incomplete information, the state uncertainty creates another dimension of priced risk. Although we don’t model uncertainty aversion directly into investors’ utility function as in Kogan and Wang (2002) and Veronesi (2001), stock returns reflect a premium in addition to the market risk premium. Moreover, the uncertainty risk premium is an increasing function of $\pi_t$, the investor’s estimate of the current state of economy, as well as $\pi_t(1 - \pi_t)$, the measure of the uncertainty about the estimate.

For reasonable range of parameters, we show that noisier firms have higher covariance with uncertainty shocks, and have higher premia. Empirically, as we discussed in the paper, this may to some extent rationalize the Fama and French (1993) size and book-to-market factors.

To conduct the empirical test, we propose two measures to proxy the state uncertainty. The first is the ASA-NBER survey of professional forecasters obtained from Federal Reserve Bank of Philadelphia. The second measure is the fitted value from a Probit model, where the dependent variable takes a value of one whenever the GDP growth rate is greater than its unconditional median, and the explanatory variables are the instruments that have been used to forecast GDP in the literature. We construct a number of variables to proxy for different aspects of uncertainty. We conduct two sets of tests of the model: the tests based on the two-pass cross-sectional regressions, as well as the tests based on the GMM estimate of the equivalent pricing kernel.

Empirical evidence is generally supportive of the model. The results suggest that uncertainty factors are priced in both sets of tests. In the cross-sectional regressions, the estimates of the premia of the uncertainty factors are significant with or without the presence of the Fama-French factors. This is consistent with a time-varying uncertainty risk premium which our model implies. The Fama-French size and book-to-market factors lack marginal explanatory power when the uncertainty factors are included. This is true for both measures of uncertainty factors: the direct measure using ASA-NBER survey data, and the fitted value measure from the Probit model.

The results based on the GMM estimates of the pricing kernel are consistent with the cross-sectional regression approach. While the estimated risk premia for the uncertainty factors are significant, Fama-French factors is significantly reduced in the presence of the uncertainty
factors, and we cannot reject the null hypothesis that both Fama-French factors are jointly zero.

To assess the economic significance of the uncertainty risk premium, we sort individual stocks into 10 portfolios based on their uncertainty betas. The average return difference among high and low uncertainty beta portfolios is about 6 percent per year, and is statistically significant. It remains significant after controlling for other factors. We then construct an uncertainty factor mimicking portfolio defined as the high minus low uncertainty beta portfolios. The estimated premium for this mimicking portfolio is about 4.4 percent per year. And after controlling for this mimicking portfolio, none of the estimated premia for the Fama-French factors as well as the momentum factor is statistically significant. All of these results are again consistent with our model’s prediction that Fama-French factors are proxies for uncertainty risk.

In summary, our results suggest that state uncertainty plays an important role in the cross-section of asset returns.
References


Appendix A. Proof of Lemma 1

From equation (26), we can write the realized excess returns as

\[ r_{i,t+1}^e = \lambda_{m,t} \beta_{m,t} + \lambda_{\pi,t} \beta_{\pi,t} + \epsilon_{i,t+1} \]  

(A1)

Take unconditional expectation on both sides of (A1), we have

\[ E r_{i,t+1}^e = \bar{\lambda}_m \bar{\beta}_m + \bar{\lambda}_\pi \bar{\beta}_\pi + \text{cov}(\beta_{im,t}, \lambda_{m,t}) + \text{cov}(\beta_{i\pi,t}, \lambda_{\pi,t}) \]  

(A2)

where \( \bar{\lambda}s \) and \( \bar{\beta}s \) are the unconditional means of the conditional \( \lambda{s} \) and \( \beta{s} \).

Define \( \frac{\text{cov}(\beta_{im,t}, \lambda_{m,t})}{\sigma_m^2} \equiv v_{im} \), and \( \frac{\text{cov}(\beta_{i\pi,t}, \lambda_{\pi,t})}{\sigma_\pi^2} \equiv v_{i\pi} \), we can rewrite

\[ \beta_{im,t} = \bar{\beta}_{im} + v_{im}(\lambda_{m,t} - \bar{\lambda}_m) + \eta_{im,t} \]  

(A3)

\[ \beta_{i\pi,t} = \bar{\beta}_{i\pi} + v_{i\pi}(\lambda_{\pi,t} - \bar{\lambda}_\pi) + \eta_{i\pi,t} \]  

(A4)

It follows that

\[ E[\eta_{im,t}] = 0, \quad E[\eta_{im,t}, \lambda_{m,t}] = 0, \]  

(A5)

and

\[ E[\eta_{i\pi,t}] = 0, \quad E[\eta_{i\pi,t}, \lambda_{\pi,t}] = 0. \]  

(A6)

Substitute equations (A3), (A4) into (A1), we have

\[ r_{i,t+1}^e = \bar{\lambda}_m \bar{\beta}_{im} + \lambda_{m,t}(\lambda_{m,t} - \bar{\lambda}_m) v_{im} + \lambda_{m,t} \eta_{im,t} + \bar{\lambda}_\pi \bar{\beta}_{i\pi} + \lambda_{\pi,t}(\lambda_{\pi,t} - \bar{\lambda}_\pi) v_{i\pi} + \lambda_{\pi,t} \eta_{i\pi,t} + \epsilon_{i,t+1} \]  

(A7)

We next assume that there are some constants \( \{j_1, j_2, j_3, j_4\} \), so that \( E[\epsilon_{i,t+1} R_{m,t+1}] = j_1 \), \( E[\epsilon_{i,t+1} \lambda_{m,t+1}] = j_2 \), \( E[\epsilon_{i,t+1} \lambda_{\pi,t+1}] = j_3 \), and \( E[\epsilon_{i,t+1} \lambda_{\pi,t+1}] = j_4 \). Similar to Jaganathan and Wang (1996), we also assume that the residual betas \( \eta_{ix,t} \) \( (s = m, \pi) \) are uncorrelated with the market return and the conditional premia. Next we define \( \{\beta_{im}, \beta_{i\pi}, \bar{\beta}_{im}, \bar{\beta}_{i\pi}\} \) as the unconditional betas, we then have:
\[
\begin{align*}
\text{cov}(r_{t+1}, r_{m,t+1}) &= \sigma^2_{m,t} \beta_{im} = k_1 + a_1 \tilde{\beta}_{im} + a_2 v_{im} + a_3 \tilde{\beta}_{i}\pi + a_4 v_{i}\pi \\
\text{cov}(r_{t+1}, \pi_{t+1}) &= \sigma^2_{\pi,t} \beta_{i}\pi = k_2 + b_1 \tilde{\beta}_{im} + b_2 v_{im} + b_3 \tilde{\beta}_{i}\pi + b_4 v_{i}\pi \\
\text{cov}(r_{t+1}, \lambda_{m,t+1}) &= \sigma^2_{\lambda,m,t} \beta_{\lambda_m} = k_3 + c_1 \tilde{\beta}_{i,m} + c_2 v_{im} + c_3 \tilde{\beta}_{i}\pi + c_4 v_{i}\pi \\
\text{cov}(r_{t+1}, \lambda_{\pi,t+1}) &= \sigma^2_{\lambda,\pi,t} \beta_{\lambda\pi} = k_4 + d_1 \tilde{\beta}_{im} + d_2 v_{im} + d_3 \tilde{\beta}_{i}\pi + d_4 v_{i}\pi
\end{align*}
\] (A8)

Assuming the coefficient matrix is non-singular, equation (A8) suggests that \( \{ \tilde{\beta}_{im}, v_{im}, \tilde{\beta}_{i}\pi, v_{i}\pi \} \) is a linear function of \( \{ \beta_{im}, \beta_{i}\pi, \beta_{\lambda_m}, \beta_{i,\lambda\pi} \} \), so that equation (A2) can then be rewritten as

\[
E[r_{i,t}^e] = p_0 + p_1 \beta_{im} + p_3 \beta_{i}\pi + p_3 \beta_{i}\lambda_m + p_4 \beta_{i,\lambda\pi}
\] (A9)

Since both \( \lambda_{mt} \) and \( \lambda_{\pi t} \) are functions of \( \pi_t \) and \( \pi_t (1 - \pi_t) \), we next apply a first order Taylor expansion on \( \lambda_{mt} \) and \( \lambda_{\pi t} \) around the unconditional mean of \( \pi_t \) and \( \pi_t (1 - \pi_t) \), it is straightforward to show that there are some constants \( \{ q_0, q_1 q_2 q_3 \} \), such that

\[
E[r_{i,t}^e] = q_0 + q_1 \beta_{im} + q_2 \beta_{i}\pi + q_3 \beta_{i,\pi}(1 - \pi)
\] (A10)

q.e.d.
Notes

1 These results suggest that expected returns are serially correlated and predictable, volatility is persistent and counter cyclical, and the relationship between expected returns and volatility is ambiguous. In the context of the consumption-based asset-pricing model of Breeden (1979), there is the famous equity premium puzzle (Mehra and Prescott (1985)), and the interest rate puzzle (Weil (1989)). In the context of the cross-sectional returns, Fama and French (1992, 1993) initiate the long-running debate about the role of beta and the explanatory power of size and book-to-market.

2 we define $\sigma_{\pi,t}^2 \equiv ||\sigma'_{\pi,t} \sigma_{\pi,t}||$, and $\sigma_{M,t}^2 \equiv ||\sigma'_{M,t} \sigma_{M,t}||$

3 Another approach is to define the entropy measure: $- [\pi \log \pi + (1 - \pi) \log (1 - \pi)]$, which also peaks at $\pi = .5$. I thank Michael Stutzer for pointing this out. It turns out this measure is highly correlated with PR_UNC, with correlation coefficient higher than .97.

4 I thank David Bates for providing me the data.

5 The sample is a little shorter than the survey data because LIM stopped updating dividend yield on S&P 500 after 1999. Applying the method of Fama and French (1989), we also constructed quarterly dividend yield on S&P 500 index using CRSP data from 1969:01 to 2001:04. The results from this expanded dataset are essentially the same to those we report in this paper.

6 I thank Kenneth French for making the data publicly available.
Following Shapiro (2002), we also used the sample covariance of the excess returns as an alternative weighting matrix, and the results are qualitatively the same as what we report here.

We also tried to use only PR_MEAN and PR_UNC as uncertainty factors. The result is qualitatively similar, and the average premium for PR_MEAN is 0.41(OLS) and 0.20(GLS), that for PR_UNC is 0.27 and 0.13, both are significant at 5% level. Fama-French factors are not statistically significant in the presence of uncertainty factors.

The right hand side variables in the second pass are estimated variables, which may cause the error-in-variable (EIV) problem. Shanken (1992) provides a correction, which is not suited for the current setting where the regressions in the first pass are univariate. Jaganathan and Wang (1996) develop another correction which also need some strong assumptions. In our case, as Fama and Macbeth (1973) argued, grouping stocks into portfolios greatly reduce the estimation noise so that the problem is mitigated. Moreover, our later GMM tests can provides a robustness check since the premia can be estimated in one pass and less distributional assumptions on the variables are needed there.

when we include only SMB and HML, the results are similar to that of Model 2. The estimated premium on the uncertainty factor mimicking portfolio is 4.1% per year, and statistically significant at 5% level. Estimates of the premia on SMB and HML are not statistically significant at 10% level.

This assumes that the part of the residuals related to the factors are not firm specific. The firm specific component has been accounted for in $\beta$s. A special case would be that the
unexpected return \( \varepsilon_{i,t+1} \) is random noise so that the unconditional correlation between \( \varepsilon_{i,t+1} \) and the above factors are zero.
Table I

Summary Statistics of the Forecasts of the Probability of Growing GDP

This table reports the characteristics of professional forecaster’s forecasts of the probability of growing GDP. The data is part of ASA-NBER survey data obtained from Federal Reserve Bank at Philadelphia. The sample is quarterly forecasts from 1969:1 to 2001:4. PR_MEAN is defined as 1 minus the mean of the forecasts of the probability of recession for the next quarter. PR_MEDIAN is the median of the same forecast. PR_MEAN3QT, is the average of the mean forecast for the current and the next two quarters. PR_UNC, the measure of uncertainty, is defined as $PR_{\text{MEAN}}(1 - PR_{\text{MEAN}})$. PR_ERR is the estimation error, measured by the cross-sectional standard deviation of individual forecasts.

<table>
<thead>
<tr>
<th># forecasters</th>
<th>PR_MEAN</th>
<th>PR_MEDIAN</th>
<th>PR_MEAN3QT</th>
<th>PR_UNC</th>
<th>PR_ERR</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>7</td>
<td>0.2594</td>
<td>0.4</td>
<td>0.2952</td>
<td>0.0211</td>
</tr>
<tr>
<td>max</td>
<td>123</td>
<td>0.9784</td>
<td>1</td>
<td>0.9760</td>
<td>0.2500</td>
</tr>
<tr>
<td>mean</td>
<td>37</td>
<td>0.7999</td>
<td>0.8333</td>
<td>0.8027</td>
<td>0.1316</td>
</tr>
<tr>
<td>median</td>
<td>34</td>
<td>0.8737</td>
<td>0.8750</td>
<td>0.8721</td>
<td>0.1104</td>
</tr>
<tr>
<td>s.t.d.</td>
<td>17</td>
<td>0.1693</td>
<td>0.1258</td>
<td>0.1625</td>
<td>0.0677</td>
</tr>
</tbody>
</table>

Correlations among PR_MEAN, PR_MEDIAN and PR_MEAN3Q

<table>
<thead>
<tr>
<th></th>
<th>PR_MEAN</th>
<th>PR_MEDIAN</th>
<th>PR_MEAN3QT</th>
</tr>
</thead>
<tbody>
<tr>
<td>PR_MEAN</td>
<td>1</td>
<td>0.9763</td>
<td>0.9769</td>
</tr>
<tr>
<td>PR_MEDIAN</td>
<td></td>
<td>1</td>
<td>0.9491</td>
</tr>
<tr>
<td>PR_MEAN3QT</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Table II
Two-pass Cross-sectional Regressions
(Uncertainty proxies constructed from the professional forecasters’ survey data)

This table reports the t-test values of the coefficients of the second step cross-sectional regression of FF25 portfolio returns on the factor $\beta_s$. The FF25 portfolio returns are quarterly excess returns of Fama-French size and book-to-market 25 portfolios from 1969:01 to 2001:04, altogether 132 quarters. The return data are downloaded from Kenneth French’s website, and are compounded quarterly. $\beta = [\beta_{\text{sMB}}, \beta_{\text{PR\_MEAN}}, \beta_{\text{PR\_UNC}}, \beta_{\text{PR\_ERR}}, \beta_{\text{SMB}}, \beta_{\text{HML}}]$ is a vector of factor loadings from the first step univariate regressions of returns on PR\_MEAN; PR\_UNC; PR\_ERR; MKT\_EX, defined as the quarterly market_excess return; SMB, defined as quarterly return difference between FF small size portfolio and large size portfolio; and HML, defined as the quarterly return difference between FF high book-to-market portfolio and low book-to-market portfolio. The last three factors are all downloaded from French’s website, and we compound the returns from monthly to quarterly. The sample is again from 1969:1 to 2001:4. $\bar{\lambda}$ is the time-series averages of loadings for each $\beta_s$. The t-values are the t-test results of the time-series of the loadings $\lambda$ for each $\beta_s$. Two models are estimated and tested. Model 1 contains only the market factor and uncertainty factors; Model 2 includes all 6 factors.

### Panel A: Model 1

<table>
<thead>
<tr>
<th>$\beta$'s</th>
<th>CONST.</th>
<th>$r_m$</th>
<th>PR_MEAN</th>
<th>PR_UNC</th>
<th>PR_ERR</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\lambda}$ (OLS)</td>
<td>0.0385</td>
<td>-0.0122</td>
<td>0.3195</td>
<td>0.2204</td>
<td>-0.0465</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{\lambda}$ (GLS)</td>
<td>0.0322</td>
<td>-0.0092</td>
<td>0.2316</td>
<td>0.153</td>
<td>-0.0366</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t-value (OLS)</td>
<td>2.8464</td>
<td>-0.8416</td>
<td>2.6446</td>
<td>2.7513</td>
<td>-1.7288</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t-value (GLS)</td>
<td>3.9573</td>
<td>-0.8278</td>
<td>2.8807</td>
<td>3.7505</td>
<td>-2.5051</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: Model 2

<table>
<thead>
<tr>
<th>$\beta$'s</th>
<th>CONST.</th>
<th>$r_m$</th>
<th>PR_MEAN</th>
<th>PR_UNC</th>
<th>PR_ERR</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\lambda}$ (OLS)</td>
<td>-0.0065</td>
<td>0.0584</td>
<td>0.3409</td>
<td>0.2181</td>
<td>-0.0467</td>
<td>-0.018</td>
<td>0.0201</td>
</tr>
<tr>
<td>$\bar{\lambda}$ (GLS)</td>
<td>0.0324</td>
<td>-0.0093</td>
<td>0.2181</td>
<td>0.145</td>
<td>-0.0343</td>
<td>0.0029</td>
<td>0.0037</td>
</tr>
<tr>
<td>t-value (OLS)</td>
<td>-0.4171</td>
<td>1.9899</td>
<td>3.4009</td>
<td>4.1871</td>
<td>-2.7607</td>
<td>-1.7442</td>
<td>1.9313</td>
</tr>
<tr>
<td>t-value (GLS)</td>
<td>2.8759</td>
<td>-0.4266</td>
<td>2.5101</td>
<td>3.2442</td>
<td>-2.3013</td>
<td>0.3489</td>
<td>0.4324</td>
</tr>
</tbody>
</table>

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Table III  
Summary Statistics of the Variables Used in the Probit Model

This table reports summary statistics for the variables used in the probit model. GDP is the GDP growth rate. DIV is the S&P 500 dividend yield. DEF is the default spread defined as the difference of quarterly yield between Moody’s BAA and BAA bond. TERM is the term spread defined as the difference of quarterly yield between 10 year government bond and 1 year government bond. TB3 is the 3 month T-bill rate. The sample is from the first quarter of 1969 through the second quarter of 1999. The dividend yield and T-bill rate data are from Logical Information Machines (LIM), and all the other rates are downloaded from the website of Federal Reserve Bank at St. Louis. All the rates are annualized and are in percentage except for autocorrelations.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Max</th>
<th>Min</th>
<th>Std. Dev.</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDPGR</td>
<td>3.036</td>
<td>3.115</td>
<td>15.411</td>
<td>-8.155</td>
<td>3.615</td>
<td>.274</td>
<td>.203</td>
<td>-.044</td>
</tr>
<tr>
<td>DIV</td>
<td>3.651</td>
<td>3.530</td>
<td>6.310</td>
<td>1.240</td>
<td>1.171</td>
<td>.937</td>
<td>.865</td>
<td>.577</td>
</tr>
<tr>
<td>DEF</td>
<td>1.105</td>
<td>0.960</td>
<td>2.610</td>
<td>0.570</td>
<td>0.443</td>
<td>.884</td>
<td>.786</td>
<td>.236</td>
</tr>
<tr>
<td>TERM</td>
<td>0.858</td>
<td>0.955</td>
<td>3.290</td>
<td>-2.140</td>
<td>1.131</td>
<td>.846</td>
<td>.740</td>
<td>-.089</td>
</tr>
<tr>
<td>TB3</td>
<td>6.646</td>
<td>6.120</td>
<td>15.550</td>
<td>2.690</td>
<td>2.614</td>
<td>.874</td>
<td>.802</td>
<td>.295</td>
</tr>
</tbody>
</table>
Table IV
Coefficient Estimates of the Probit Model and Properties of Uncertainty Proxies
Constructed from the Fitted Values

Panel A reports the estimation results of the Probit Model \( \text{Prob}(\text{state} = 1 | x) = \phi(\beta'x) \) where \( \phi(\cdot) \) represents the cumulative distribution function for the standard normal distribution. Here we set \( \text{state} = 1 \) whenever GDP growth rate is greater than its unconditional median, and zero otherwise. Vector \( x \) include DIV, DEF, TERM, and TB3, which are all defined in Table III. The data source and sample period are the same as in Table 3. “(-1)” stands for lag one. Panel B reports the descriptive statistics of the uncertainty factors constructed from Probit regression reported in Panel A of this table. Each quarter, we compute the fitted value using estimates from Panel A. Let \( \hat{\text{state}} \) be the fitted value of the Probit regression for each quarter. We then define \( PR_{-\text{MEAN}} \equiv \hat{\text{state}} \), and \( PR_{-\text{UNC}} \equiv \hat{\text{state}} \cdot (1 - \hat{\text{state}}) \). The sample spans the period from 1969:01 to 1999:02.

### Panel A: Coefficient Estimates of the Probit Model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std.err.</th>
<th>z-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-1.0387</td>
<td>0.5264</td>
<td>-1.9730</td>
<td>0.0485</td>
</tr>
<tr>
<td>GDPGR(-1)</td>
<td>0.0791</td>
<td>0.0359</td>
<td>2.2019</td>
<td>0.0277</td>
</tr>
<tr>
<td>DIV(-1)</td>
<td>-0.2899</td>
<td>0.1760</td>
<td>-1.6470</td>
<td>0.0996</td>
</tr>
<tr>
<td>DEF(-1)</td>
<td>0.2321</td>
<td>0.4049</td>
<td>0.5732</td>
<td>0.5665</td>
</tr>
<tr>
<td>TERM(-1)</td>
<td>0.5146</td>
<td>0.1423</td>
<td>3.6166</td>
<td>0.0003</td>
</tr>
<tr>
<td>TB3(-1)</td>
<td>0.1703</td>
<td>0.0813</td>
<td>2.0946</td>
<td>0.0362</td>
</tr>
<tr>
<td>LR statistic (5 df)</td>
<td>21.9052</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-72.9141</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>McFadden R-Squared</td>
<td>0.13060</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: Properties of Uncertainty Risk Proxies Constructed from the Fitted Values

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Max</th>
<th>Min</th>
<th>Std. Dev.</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PR_MEAN</td>
<td>0.496</td>
<td>0.512</td>
<td>0.895</td>
<td>0.026</td>
<td>0.206</td>
<td>.699</td>
<td>.598</td>
<td>.001</td>
</tr>
<tr>
<td>PR_UNC</td>
<td>0.208</td>
<td>0.232</td>
<td>0.250</td>
<td>0.025</td>
<td>0.050</td>
<td>.509</td>
<td>.410</td>
<td>-.215</td>
</tr>
</tbody>
</table>
Table V
Two-pass Cross-sectional Regressions
(Uncertainty proxies constructed from the Probit model)

This table reports the t-test values of the coefficients of the second step cross-sectional regression of FF25 portfolio returns on the factor $\beta_s$. The FF25 portfolio returns are quarterly excess returns of Fama-French size and book-to-market 25 portfolios from 1969:1 to 1999:2, altogether 122 quarters. The return data are downloaded from Kenneth French’s website, and are compounded quarterly. $\beta = [\beta_{r_s}, \beta_{PR\_MEAN}, \beta_{PR\_UNC}, \beta_{SMB}, \beta_{HML}]$ is a vector of factor loadings from the first step univariate regressions of returns on PR_MEAN; PR_UNC; MKT_EX, defined as the quarterly market excess return; SMB, defined as quarterly return difference between FF small size portfolio and large size portfolio; and HML, defined as the quarterly return difference between FF high book-to-market portfolio and low book-to-market portfolio. PR_MEAN and PR_UNC are constructed from Table V. The other three factors are all downloaded from French’s website, and we compound the returns from monthly to quarterly. The sample is again from 1969:01 to 1999:02. $\lambda$ is the time-series averages of loadings for each $\beta$. The t-values are the t-test results of the time-series of the loadings $\lambda$ for each $\beta$. Two models are estimated and tested. Model 1 contains only the market factor and uncertainty factors; Model 2 includes all 5 factors.

<table>
<thead>
<tr>
<th>Panel A: Model 1</th>
<th>loadings on $\beta_s$</th>
<th>CONST.</th>
<th>$r_m$</th>
<th>PR_MEAN</th>
<th>PR_UNC</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ (OLS)</td>
<td>0.0047</td>
<td>0.0146</td>
<td>0.2431</td>
<td>-0.0240</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$ (GLS)</td>
<td>0.0215</td>
<td>0.0012</td>
<td>0.1567</td>
<td>-0.0311</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t-value (OLS)</td>
<td>0.3140</td>
<td>0.8304</td>
<td>3.1218</td>
<td>-1.3736</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t-value (GLS)</td>
<td>2.3693</td>
<td>0.0988</td>
<td>3.9879</td>
<td>-3.2010</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Model 2</th>
<th>loadings on $\beta_s$</th>
<th>CONST.</th>
<th>$r_m$</th>
<th>PR_MEAN</th>
<th>PR_UNC</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ (OLS)</td>
<td>-0.0305</td>
<td>0.0702</td>
<td>0.2082</td>
<td>-0.0040</td>
<td>-0.0158</td>
<td>0.0191</td>
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<tr>
<td>$\lambda$ (GLS)</td>
<td>0.0257</td>
<td>-0.0076</td>
<td>0.1592</td>
<td>-0.0328</td>
<td>0.0035</td>
<td>-0.0035</td>
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<tr>
<td>t-value (OLS)</td>
<td>-1.6293</td>
<td>2.3315</td>
<td>4.0601</td>
<td>-0.2680</td>
<td>-1.5345</td>
<td>1.9464</td>
<td></td>
</tr>
<tr>
<td>t-value (GLS)</td>
<td>2.0576</td>
<td>-0.3618</td>
<td>3.8834</td>
<td>-2.9971</td>
<td>0.4297</td>
<td>-0.4442</td>
<td></td>
</tr>
</tbody>
</table>

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**Table VI**  
GMM Estimates of the Reduced-form Uncertainty Pricing Kernel

This table reports GMM estimates of the pricing kernel model $E(m_{t+1}r_{t+1})=0$, where, $m_{t+1} = 1 + \beta f_{t+1}$ and $f_{t+1}$ includes the market excess return $r_m$, Fama-French factors SMB and HML, and the demeaned uncertainty factors PR_MEAN, PR_UNC, and PR_ERR. $r_{t+1}$ is the quarterly excess returns of Fama-French 25 size and book-to-market portfolios from 1969:01 to 2001:04, altogether 132 quarters. The return data are downloaded from Kenneth French’s website. We consider four scenarios: Model 1 contains only 3 FF factors, Model 2 contains only the market factor and PR_MEAN, and Model 3 includes the market factor and all uncertainty factors, and Model 4 contains all factors. For each case, we report the coefficients, $p$-values as well as the $J$-statistics. We use Newey-West adjusted covariance matrices to account for potential auto-correlation. We also report the results of the Wald tests of the hypotheses whether the Fama-French factors or the uncertainty factors are jointly zero.

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th></th>
<th>Model 2</th>
<th></th>
<th>Model 3</th>
<th></th>
<th>Model 4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coef.</td>
<td>$p$-value</td>
<td>Coef.</td>
<td>$p$-value</td>
<td>Coef.</td>
<td>$p$-value</td>
<td>Coef.</td>
<td>$p$-value</td>
</tr>
<tr>
<td>$\beta_{r_m}$</td>
<td>-7.2901</td>
<td>0.0000</td>
<td>-1.6421</td>
<td>0.0000</td>
<td>-1.9446</td>
<td>0.0001</td>
<td>-6.3288</td>
<td>0.0009</td>
</tr>
<tr>
<td>$\beta_{PR_MEAN}$</td>
<td>-9.4238</td>
<td>0.0000</td>
<td>-9.4809</td>
<td>0.0000</td>
<td>-14.3081</td>
<td>0.1237</td>
<td>-78.4117</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\beta_{PR_UNC}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{PR_ERR}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{SMB}$</td>
<td>3.3273</td>
<td>0.0000</td>
<td>2.5426</td>
<td>0.0000</td>
<td>4.0409</td>
<td>0.0683</td>
<td>34.1337</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\beta_{HML}$</td>
<td>-4.9547</td>
<td>0.0000</td>
<td>-0.5554</td>
<td>0.2376</td>
<td>-3.01220</td>
<td>0.1172</td>
<td>12.6837</td>
<td>(0.9193)</td>
</tr>
<tr>
<td>$J$-statistic</td>
<td>20.2181</td>
<td>(0.5694)</td>
<td>17.8692</td>
<td>(0.7653)</td>
<td>17.7068</td>
<td>(0.6674)</td>
<td>12.6837</td>
<td>(0.9193)</td>
</tr>
</tbody>
</table>

Wald Tests:

- $H_0: \beta_{SMB} = \beta_{HML} = 0$
  $\chi^2 = 100.68$
  $p$-value = 0.00

- $H_0: \beta_{SMB} = \beta_{HML} = 0$
  $\chi^2 = 19.72$
  $p$-value = 0.0001

- $H_0: \beta_{SMB} = \beta_{HML} = 0$
  $\chi^2 = 4.2786$
  $p$-value = 0.1177
Table VII
Average Returns and Alphas of Uncertainty Beta Sorted Portfolios

Starting from December 1973 to December 2001, at the end of each year, we estimate the uncertainty betas for all the common stocks traded at NYSE, AMEX and NASDAQ using the quarterly data for the past 5 years. We define the uncertainty beta as the coefficients of the PR_MEAN in the multi-variate regression of excess returns on the market factor and PR_MEAN. We then rank the uncertainty betas and sort the stocks into 10 portfolios based on their rankings of the betas. Portfolio 1 consists of firms with smallest uncertainty betas, and portfolio 10 consists of those with the highest betas. We record each portfolio’s monthly equally-weighted returns and obtain a return series of 336 months for each portfolio. This table reports the α for each portfolio, defined as the intercepts from the regressions of portfolio excess returns on (a) constant, (b) market excess returns, and (c) the four factors (FF three factors plus the momentum factor). We also report the p-value of the hypothesis Ho: alpha10-alpha1 = 0, and the F-statistic (and its p-value in the parentheses) of the GRS test of Ho: all the 10 alphas = 0. The monthly returns for the Fama-French factors SMB, HML and the momentum factor are downloaded from Kenneth French’s website. All returns are in percentage.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>p-value</th>
<th>GRS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td>(10-1)</td>
<td>(p-value)</td>
</tr>
<tr>
<td>Constant</td>
<td>alpha</td>
<td>0.8886</td>
<td>1.0121</td>
<td>0.9370</td>
<td>1.0485</td>
<td>1.0692</td>
<td>1.1676</td>
<td>1.0868</td>
<td>1.1675</td>
<td>1.2629</td>
<td>1.3934</td>
<td>.0098</td>
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<tr>
<td>Market</td>
<td>alpha</td>
<td>0.2051</td>
<td>0.4213</td>
<td>0.3981</td>
<td>0.5319</td>
<td>0.5708</td>
<td>0.6436</td>
<td>0.5566</td>
<td>0.6160</td>
<td>0.6659</td>
<td>0.7001</td>
<td>.0121</td>
</tr>
<tr>
<td></td>
<td>t-value</td>
<td>0.8473</td>
<td>2.6251</td>
<td>2.9077</td>
<td>3.9732</td>
<td>4.4357</td>
<td>4.5195</td>
<td>3.7751</td>
<td>3.5572</td>
<td>3.1640</td>
<td>2.3807</td>
<td>(0.0017)</td>
</tr>
<tr>
<td>4 Factors</td>
<td>alpha</td>
<td>0.2035</td>
<td>0.2315</td>
<td>0.1334</td>
<td>0.2350</td>
<td>0.2735</td>
<td>0.3625</td>
<td>0.3133</td>
<td>0.3798</td>
<td>0.4292</td>
<td>0.6049</td>
<td>.0357</td>
</tr>
<tr>
<td></td>
<td>t-value</td>
<td>1.2698</td>
<td>2.4363</td>
<td>1.6567</td>
<td>3.2673</td>
<td>3.9927</td>
<td>4.9919</td>
<td>4.2358</td>
<td>4.2099</td>
<td>3.7207</td>
<td>3.1314</td>
<td>(0.0009)</td>
</tr>
</tbody>
</table>
Table VIII
Estimating Uncertainty Risk Premia Using Factor Mimicking Portfolio(s)

This table reports the results of the two-pass cross-sectional regressions. In the first pass, for each month from September 1973 to December 2001, we regress the Fama-French size and book-to-market 25 portfolio excess returns on the market excess returns, the returns of Fama-French factors (SMB and HML) plus the momentum factor (UMD), and the uncertainty mimicking factor returns UNC (defined as the difference between the average return of the three highest uncertainty beta portfolios and the average return of the three lowest uncertainty beta portfolios. Panel A includes one uncertainty factor mimicking portfolio; Panel B considers two factor mimicking portfolios (UNC1 for PR_MEAN and UNC2 for PR_UNC). The coefficients are defined as factor betas. Each month we update the factor betas using a three-year rolling window. In the second pass, for each month we run the cross-sectional regression of portfolio excess returns on the estimated factor betas. The coefficients are the estimated factor risk premia $\lambda$ s. We report the time-series average $\bar{\lambda}$ for each factor, and $p$-value as well as the Shanken (1992) EIV corrected $p$-value for the null hypothesis that $\bar{\lambda} = 0$. We estimate and test four models. In model 1 and model 3 we include only the uncertainty factor(s) and the market excess return. In model 2 and model 4 we also add the Fama-French factors SMB and HML, and a momentum factor UMD. The monthly returns for the Fama-French factors SMB, HML, UMD and the returns for 25 size and book-to-market portfolios are all downloaded from Kenneth French’s website. All estimated premia are in percentage.

### Panel A

<table>
<thead>
<tr>
<th></th>
<th>Intercept</th>
<th>$\beta_{\text{MKT}}$</th>
<th>$\beta_{\text{UNC1}}$</th>
<th>$\beta_{\text{UNC2}}$</th>
<th>$\beta_{\text{SMB}}$</th>
<th>$\beta_{\text{HML}}$</th>
<th>$\beta_{\text{UMD}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model 1</strong></td>
<td>$\bar{\lambda}$</td>
<td>0.9959</td>
<td>-0.4141</td>
<td>0.4622</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$p$-value</td>
<td>0.0045</td>
<td>0.2932</td>
<td>0.0264</td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Shanken $p$-value</td>
<td>0.0053</td>
<td>0.2983</td>
<td>0.0280</td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Model 2</strong></td>
<td>$\bar{\lambda}$</td>
<td>1.0669</td>
<td>-0.4468</td>
<td>0.3747</td>
<td>0.1704</td>
<td>0.3008</td>
<td>0.0719</td>
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<tr>
<td></td>
<td>$p$-value</td>
<td>0.0003</td>
<td>0.1609</td>
<td>0.0198</td>
<td>0.3663</td>
<td>0.1111</td>
<td>0.8065</td>
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<tr>
<td></td>
<td>Shanken $p$-value</td>
<td>0.0003</td>
<td>0.1620</td>
<td>0.0201</td>
<td>0.3648</td>
<td>0.1091</td>
<td>0.8076</td>
</tr>
</tbody>
</table>

### Panel B

<table>
<thead>
<tr>
<th></th>
<th>Intercept</th>
<th>$\beta_{\text{MKT}}$</th>
<th>$\beta_{\text{UNC1}}$</th>
<th>$\beta_{\text{UNC2}}$</th>
<th>$\beta_{\text{SMB}}$</th>
<th>$\beta_{\text{HML}}$</th>
<th>$\beta_{\text{UMD}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model 3</strong></td>
<td>$\bar{\lambda}$</td>
<td>1.1221</td>
<td>-0.4637</td>
<td>0.6086</td>
<td>0.5534</td>
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<td></td>
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<tr>
<td></td>
<td>$p$-value</td>
<td>0.0003</td>
<td>0.1856</td>
<td>0.0018</td>
<td>0.0025</td>
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<td></td>
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<td>Shanken $p$-value</td>
<td>0.0004</td>
<td>0.1894</td>
<td>0.0017</td>
<td>0.0025</td>
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<td></td>
</tr>
<tr>
<td><strong>Model 4</strong></td>
<td>$\bar{\lambda}$</td>
<td>1.0239</td>
<td>-0.3973</td>
<td>0.4331</td>
<td>0.3142</td>
<td>0.1764</td>
<td>0.3058</td>
</tr>
<tr>
<td></td>
<td>$p$-value</td>
<td>0.0005</td>
<td>0.2039</td>
<td>0.0072</td>
<td>0.0480</td>
<td>0.3504</td>
<td>0.1042</td>
</tr>
<tr>
<td></td>
<td>Shanken $p$-value</td>
<td>0.0006</td>
<td>0.2060</td>
<td>0.0067</td>
<td>0.0472</td>
<td>0.3489</td>
<td>0.1020</td>
</tr>
</tbody>
</table>

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